UNIVERSITÀ DI PISA

Corso di Laurea Triennale in Matematica

Tesi di Laurea

Deformation theory and local study of the Hilbert scheme

Relatore:
Prof. Mattia Talpo

Candidato:
Luca Morstabilini

Anno Accademico 2020/21

Dla Janusza,
i jego matematyczne zagadki o ptakach,
i dla Joli,
która zawsze opiekowała się mną.

Contents

Introduction

A central problem in algebraic geometry is that of classification and, closely related to that, understanding how objects vary in families. For example, one natural question to ask is wheter there exists the "space of all smooth curves" and, if it exists, what are its properties. This is a problem dating as far back as Riemann's works on Riemann surfaces, when he proved that a Riemann surface of genus g with $g \geq 2$ depends on $3g-3$ parameters, or, in other terms, the "space of genus g curves" is $3g-3$-dimensional, if $g \geq 2$. One can, of course, ask the same question for more complicated objects, such as sheaves, closed subschemes, morphisms... One notable example of such a classifying space is the Hilbert scheme, a scheme parametrizing the closed subschemes of a projective scheme Y, whose local study will be the guideline of this work through the means of deformation theory.
The object of study of deformation theory is the study of the local properties of families of objects. This is related to the previous problem, in the sense that, when a classifying space as above exists, called a *moduli space*, then deformation theory can be used to provide information about said space.
In this work all schemes will be k-schemes, for a fixed algebraically closed field k. A deformation of a scheme X will be the datum of a flat and proper morphism of schemes $\mathcal{X} \to S$ with a fixed rational point $s \in S$ such that the fiber over s is isomorphic to the scheme X we started with. Since we want to apply the methods of deformation theory to the study of Hilbert schemes, we will instead consider

deformations as above that, roughly, preserve the structure of X as a closed subscheme.
The study of deformations of an object is done in various subsequent steps: one first studies the so called *infinitesimal deformations*, which are deformations over Artinian rings. These are deformations of our object X obtained by adjoining some nilpotent parameters which result in X when the parameters vanish. So, for example, if X is the vanishing locus of some set of polynomials $\{f_i\}_i$, an example of deformation of X could be obtained by considering the vanishing locus of $\{f_i + \varepsilon g_i\}_i$, with $\varepsilon^2 = 0$. The use of nilpotents shows that, for the study of deformation theory, we need the language of schemes: classical algebraic geometry cannot handle well nilpotents.
In the study of infinitesimal deformations, one first studies the *first order deformations*, which are, roughly speaking, the deformations obtained by adjoining a single parameter of square 0. In complex algebraic geometry, this would be akin to considering the first-order Taylor expansion of the equations defining our object.
To study higher-order deformations, one important topic is that of obstructions. Given a surjection of Artinian rings $A' \to A$ with kernel isomorphic to k, one can ask if a given deformation defined over A can be lifted to a deformation defined over A'. This problem is related to the vanishing of a certain element, the *obstruction to the lifting*, of a vector space over k.
The study of higher order deformations then leads to formal deformations, which are collections of compatible infinitesimal deformations over all quotients $R/\mathfrak{m}_R^n$, where R is a complete Noetherian ring such that $R/\mathfrak{m}$ is k. Finally, we consider the problem of algebraizability, that is, if a given formal deformation is induced by a deformation of X over an scheme of finite type.
In this work, we will only consider in detail the case of first order deformations,

we will define formal deformations and introduce the important concepts of various kind of *universality* properties, and will not touch upon the problem of algebraizability.
There are various ways to study of the deformation theory of inifnitesimal deformations. One, the one used in this work, is due to Schlessinger, who reformulated deformation theory in the language of functors from categories of local Artinian rings to the category of sets. Given such a functor F, we will require that $F(k)$ be a singleton, and we will consider elements of $F(A)$, for an Artinian ring A with residue field k, as the deformations of $* \in F(k)$. All the notions relevant in deformation theory will be given for such functors, and we will then specialize them to the deformation problem of closed subschemes. The pinnacle of Schlessinger's approach is the Schlessinger theorem, which will be the arriving point of the last chapter

The outline of this thesis is as follows. In chapter 1 we introduce all the the notions that will be later used: functors of Artinian rings, deformations and, quickly, moduli spaces. We define and state the existence of the Hilbert scheme, the moduli space we will study with the help of deformation techniques.
In chapter 2 we will show that, for suitable functors of Artin rings F, there is a notion of space tangent to F. We will then compute the tangent space to the functor of deformations of closed subschemes, and then show how this tangent space is indeed of geometric nature, showing its relationship with the Hilber scheme.
The theme of chapter 3 is that of liftings. Given a surjection of Artinain rings $A' \to A$, and a deformation over A, one can ask if this deformations lifts to one over A'. We will show that this question is related to vanishing of an element of a vector space, the *obstruction* to the lifting. We will then show how the problem of lifting deformation is connected with geometry, and as a consequence we will

deduce the smoothness of various Hilbert schemes.
In chapter 4, we introduce the notion of formal elements and of formal deformations. We define what universal and miniversal deformations are, and state the Schlessinger theorem, a fundamental result that determines whether (mini)versal deformations exist for a given deformation problem.

All chapters are organized as follows: in the first section we develop the abstract deformation theory using functors of Artin rings. In the second section we specialize the notion introduced in the first to the problem of deformations of closed subschemes. In the last section of a chapter (except for the last chapter) we apply the work done previously to do some geometry.

Notations and conventions

Throughout the text, k will denote a fixed algebraically closed field. All schemes will be k-schemes, and all the morphisms will be assumed to be morphisms of k-schemes. We will use Λ to denote a complete Noetherian local ring with residue field k.
By an algebraic scheme we will mean a k-scheme $(X, \mathcal{O}_X)$ of finite type. If there will be no confusion about the scheme structure on $(X, \mathcal{O}_X)$, we will simply denote it by X, forgetting the structure sheaf.
If S is a k-scheme and $s \in S$ a closed point, we will denote the morphisms from $\mathrm{Spec}(k)$ to S having image s by $s : \mathrm{Spec}(k) \to S$.
Given a morphism $f : X \to Y$ of of finite type and a point $y \in Y$, we will use the notation $X(y)$ to denote the fiber scheme of f along y, that is $X(y) = X \times_Y \mathrm{Spec}(k(y))$.
Given a coherent sheaf $\mathcal{F}$ on a projective scheme X, $h^i(X, \mathcal{F})$ will denote the k-dimension of $H^i(X, \mathcal{F})$.

Given a category $\mathcal{C}$, the notation $X \in \mathcal{C}$ will mean that X is an object of $\mathcal{C}$. The terms *map* and *morphism* will be synonymous.

Chapter 1

The main tools

In this chapter we introduce the main tools we will be using. We define the functors of Artin rings, and we show that the categories of local Artinian rings we will work with have fibered products. We then give a definition for the deformations of a scheme, both for "abstract" schemes and for embedded schemes. The latter type will be the deformations we will use to study the local geometry of the Hilbert scheme, a scheme that we define by its functor of points and of which we state the existence in the third part of the chapter.

1.1 Functors of Artin Rings

Definition 1.1.1. *The category $\mathcal{A}_\Lambda$ is the category having local Artinian Λ-algebras with residue field k as objects and morphisms are local Λ-algebras morphisms.*

If $\Lambda = k$, we will simply denote $\mathcal{A}_k$ by $\mathcal{A}$. The most important object of $\mathcal{A}$ will be the ring of dual numbers $k[x]/(x^2)$. We will denote it by $k[\varepsilon]$.

Definition 1.1.2. *A functor of Artin rings is a functor $F : \mathcal{A}_\Lambda \to Set$. A morphism of functors of Artin rings is just a natural transformation between the functors.*

Example 1.1.1. The simplest and most important examples of functors of Artin rings are the following. Let R a complete Λ-algebra with residue field k: then $\mathrm{Hom}_\Lambda(R, -) : \mathcal{A}_\Lambda \to \mathrm{Set}$ is a functor. Functors isomorphic to one of this kind are called *prorepresentable*, and are said to be prorepresented by R.

We will use the following important property:

Proposition 1.1.3. *The category $\mathcal{A}_\Lambda$ has fibered products.*

Proof. Suppose we are given three rings A, A', $A'' \in \mathcal{A}_\Lambda$, and morphisms $\phi : A' \to A$, $\psi : A'' \to A$. The set $A' \times_A A'' = \{(a', a'')|\phi(a') = \psi(a'')\}$ inherits a ring structure from the the ones on A' and A'', by pointwise product. Morevoer, it is the fibered product of A' and A'' over A in the category of rings. It clearly also remains a Λ-algebra. It only remains to show it is local Artinian and with residue field k. Since it is a submodule of $A' \times A''$, which has finite length as Λ-module, $A' \times_A A''$ has finite length, too. So, we conclude that it is Artinian. Consider the ideal $\mathfrak{m}$, consisting of the elements $(a', a'') \in A' \times_A A''$ such that $a' \in \mathfrak{m}_{A'}$ and $a'' \in \mathfrak{m}_{A''}$. It is proper, and we now show it is the unique maximal ideal. Indeed, suppose (a', a'') is not an element of $\mathfrak{m}$. Then, without loss of generality, a' is a unit in A'. Since $\psi(a'') = \phi(a')$ and the morphisms are local, a'' must be a unit in A'', too. Indeed, $\psi(a'')$ is a unit, and by locality of ψ non-units in A'' are sent to non-units in A. So there are inverses b', b'' respectively for a' and a''. It is immediate to see that $(b', b'') \in A' \times_A A''$ and that is an inverse for (a', a'') in said ring.
Finally, all the rings considered are also k-algebras, that is, there is an embedding $k \to A$ such that projection onto the residue field is the identity of k, and similarly for A', A''. So, given $x \in k$, the element $(x, x) \in A' \times_A A''$ is mapped to x via the composition $A' \times_A A'' \to A' \to k$. Hence k is a quotient of $A' \times_A A''$. □

Remark 1.1.1. The existence of fibered products has the following consequence. Given A, A', A'' as above and a functor of Artinian rings F, we have the following commutative diagram, induced by functoriality of F.

$$\begin{array}{ccc} F(A' \times_A A'') & \longrightarrow & F(A') \\ \downarrow & & \downarrow \\ F(A'') & \longrightarrow & F(A) \end{array}$$

Hence, we get a unique map $\alpha : F(A' \times_A A'') \to F(A') \times_{F(A)} F(A'')$.

In the following chapters we will use the map α to determine properties of the functor F, and to do so the following definition will be crucial.

Definition 1.1.4. *A small Λ-extension is a surjective morphism $\phi : A \to A'$ in $\mathcal{A}_\Lambda$ such that the kernel of ϕ has length* 1. *An extension is said to be trivial if it admits a splitting, i.e. a morphism $\psi : A \to A'$ such that $\phi \cdot \psi = id_A$.*

Remark 1.1.2. Since all ideals of an Artinian ring are nilpotent, every surjection of Artin rings can be factored as a composition of small extensions.

Remark 1.1.3. The request that the kernel I be of length 1 is equivalent to ask that it be isomorphic to k, since the quotient field of Λ is k. Moreover, I is nilpotent, since A' is Artinian, and I^2 is a submodule of I. It can't be equal to I by nilpotence, so by simpleness of I it must be of square zero.

Example 1.1.2. A trivial extension can be constructed in the following way. Consider the A-module $A \oplus k$, and define a product by the rule $(a, x) \cdot (b, y) = (ab, ay + bx)$. Standard computations show that it endows $A \oplus k$ with a ring structure, and we denote it by $A{*}k$ Moreover, the canonical inclusion $A \to A \oplus k$ endows $A{*}k$ with a structure of A-algebra compatible with the module structure. The projection $(a, x) \mapsto a$ is a surjective morphism of Λ-algebras, with kernel isomorphic to k, hence small, with a clear splitting.

Given two small extensions $\phi : A'' \to A$ and $\psi : A' \to A$ of the same ring A, a morphism between them is a morphism $\chi : A'' \to A'$ such that the following is a commutative diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & k & \longrightarrow & A' & \xrightarrow{\psi} & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle id_k} & & \downarrow{\scriptstyle \chi} & & \downarrow{\scriptstyle id_A} & & \\ 0 & \longrightarrow & k & \longrightarrow & A'' & \xrightarrow{\phi} & A & \longrightarrow & 0 \end{array}$$

By the $5-$lemma, this implies that such a χ would be an isomorphism, hence all morphisms are isomorphisms.

Definition 1.1.5. *Given $A \in \mathcal{A}_\Lambda$, we denote by $o(A/\Lambda)$ the set of isomorphism classes of small Λ-extensions of A.*

Given a morphism $f : A \to B$ in $\mathcal{A}_\Lambda$ and a small extension B' of B, there is a way to obtain an extension of A. Indeed, consider the following diagram:

$$\begin{array}{ccccccccc} & & & & B' \times_B A & \xrightarrow{\psi} & A & & \\ & & & \overset{i}{\nearrow} & \downarrow & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & k & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

where ψ is the projection from the fibered product. Given $a \in A$, if $b' \in B'$ is a preimage of $f(a)$, then ψ maps (b', a) to a, so that ψ is surjective. Since the kernel of ϕ is k, the elements of $\ker \psi$ are of the form $(i(x), 0)$ with $x \in k$. Hence, we identify the kernel of ψ with k, and we see that $\psi : B' \times_B A \to A$ is a small extension of A. The extension we have just built is called *pullback* of the extension $B' \to B$ along f.

Similarly, given a morphism $g : k \to k$ and a small extension

$$0 \longrightarrow k \xrightarrow{i} A' \xrightarrow{\psi} A \longrightarrow 0$$

we can define a pushout extension in the following way. Define $A \sqcup_k k$ to be the A-algebra $A{*}k/I$, where $I = ((i(x), -g(x))$. We have the following diagram

$$\begin{tikzcd}
0 \arrow[r] & k \arrow[r, "i"] \arrow[d] & A' \arrow[r, "\psi"] \arrow[d] & A \arrow[r] \arrow[d, "id_A"] & 0 \\
& k \arrow[r] & A \sqcup_k k \arrow[r, "\chi"] & A &
\end{tikzcd}$$

The bottom row defines a small extension of A, which we call the *pushforward* of the original extension along g.

Both for the pullback and the pushforward of an extension, it is immediate to see that the isomorphism class of the construction depends only on the isomorphism class of the extension we started with. We show it for the pullback, and the argument for the pushforward case is similar.

If $B' \to B$ and $B'' \to B$ are equivalent small extensions, then by definition there is a morhpism $\xi : B' \to B''$ making the diagram

$$\begin{tikzcd}
0 \arrow[r] & k \arrow[r, "i"] \arrow[d] & B' \arrow[r, "\phi"] \arrow[d, "\xi"] & B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & k \arrow[r, "j"] & B'' \arrow[r, "\psi"] & B \arrow[r] & 0
\end{tikzcd}$$

commute.

We have to check that there is a morphism μ making the following diagram commute

$$\begin{tikzcd}
0 \arrow[r] & k \arrow[r] \arrow[d] & B' \times_B A \arrow[r] \arrow[d, dashed, "\mu"] & A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & k \arrow[r] & B'' \times_B A \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

We define $\mu(b', a) = (\phi(b'), a)$. Then, since $\mu(i(x)) = j(x)$, the first square commutes. Since $\psi(\xi(b')) = \phi(b')$, the second square commutes.

Finally, we can define a sum on $o(A/\Lambda)$. Given two estensions $\phi : A' \to A$ and $\psi : A'' \to A$, let $\phi + \psi$ be defined as follows. Consider the fibered product $A' \times_A A''$: the induced morpshim $A' \times_A A'' \to A$ is not a small extension, since the kernel has length 2, and is hence isomorphic to $k \oplus k$. Let I be the kernel of the sum morphism $\delta : k \oplus k \to k$, $(a, b) \to a + b$. Then, $(A' \times_A A'')/I$ is a small extension of A. Indeed, the ideal $(k \oplus k)/I \subset (A' \times_A A'')/I$ has length 1, and when we quotient by it we end up with A. We define $\phi + \psi$ to be this small extension.

Now, using the notion of pushout of small extensions, we can endow $o(A/\Lambda)$ with a structure of A-module. Indeed, given $a \in A$, let $g : k \to k$ by the multiplication by a. Then, given (the class of) an extension $\eta \in o(A/\Lambda)$, define $a \cdot \eta$ to be the (class of the) pushforward of η along g. With arguments similar to the one done above for the pushout, one can then show that both the sum and the multiplication are independent of the class of η, so we indeed get a structure of A-module. In particular, the action of the maximal ideal of A is trivial, because it acts trivially on k and the taking the pushforward along the trivial map yields the trivial extension $A * k$. So, the action descends to an action of k and we get the following:

Proposition 1.1.6. *The set $o(A/\Lambda)$ has a structure of A-module. In particular, it is also a k-vector space.*

1.2 Infinitesimal deformations of schemes

We start by fixing an algebraic scheme X over k. This will be the scheme we deform.

Definition 1.2.1. *Let S a connected k-scheme. A family of deformations of X by S is the datum of a cartesian diagram*

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d] & \mathcal{X} \arrow[d, "\pi"] \\ \operatorname{Spec} k \arrow[r, "s"] & S \end{tikzcd}$$

with π a proper flat surjective morphism. S is called the parameter space and $\mathcal{X}$ the total space of the deformation.

In the following, we will often refer to a family of deformations of X over S simply by calling it a deformation of X over S. If we want to specify the parameter space and the closed point s, we will use the notation (S, s, π) to denote the deformation.

The condition that the diagram is a pullback can be rephrased by asking that the fiber of π along the closed point $s \in S$ be isomorphic to the scheme X. Then the inclusion j is a closed immersion. We will call such fiber the *distinguished fiber*.

Definition 1.2.2. *Given two families of deformations $\pi : \mathcal{X} \to S$ and $\eta : \mathcal{Y} \to S$, a morphism between the deformations is a map $f : \mathcal{X} \to \mathcal{Y}$ such that the following diagram* *commutes:*

$$\begin{tikzcd}
 & X \arrow[dl] \arrow[dr] & \\
\mathcal{X} \arrow[rr, "f"] \arrow[dr, "\pi"'] & & \mathcal{Y} \arrow[dl, "\eta"] \\
 & S &
\end{tikzcd}$$

Given a family of deformations of X $\pi : \mathcal{X} \to S$, the fibers $\mathcal{X}(t)$ are also called deformations of X, for all $t \in S$ closed points. The deformations of X will all be equidimensional by flatness of π, but they need not be all isomorphic. In the same spirit, there can be deformations of a scheme X such that all fibers are isomorphic to X, but the family is not trivial.

Example 1.2.1. Consider $\mathcal{X} = X \times S$ and let $\pi : \mathcal{X} \to S$ the projection map. Deformations isomorphic to one of this kind are called *trivial*.

Example 1.2.2. Consider $S = \mathbb{A}^1$, $\mathcal{X} = \operatorname{Spec} k[x,y,t](xy-t)$ and $\pi : \mathcal{X} \to S$ the morphism associated to the inclusion $k[t] \hookrightarrow k[x,y,t]/(xy-t)$. It is readily verified that it is surjective, and since the source is Noetherian reduced and the target is a Dedekind scheme, flatness follows from lemma [Har77, Chapter III, Prop. 9.7]. Then $X_0 = \mathcal{X}(0)$ has a singularity at the origin, while all the other fibers are smooth and isomorphic to each other.

Example 1.2.3. A ruled surface is a nonsingular projective surface X together with a morphism π to a nonsingular projective curve C whose fibers are copies of $\mathbb{P}^1$ and admitting a section $\sigma : C \to X$. Hence, a ruled surface defines a deformation of $\mathbb{P}^1$ over C. This deformations have all isomorphic fibers but, however, there exist ruled surfaces that are not trivial, that is such that X is not isomorphic to $C \times \mathbb{P}^1$. One such example are the Hirzebruch surfaces $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(n))$, which are not isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ if $n \neq 0$.

Given a deformation (S, s, π), a morphism $g : S' \to S$ and a closed point $s' \in S'$ mapping to s, we can consider the diagram

$$\begin{tikzcd} X \arrow[r, "j'"] \arrow[d] & \mathcal{X} \times_S S' \arrow[d, "\pi_{S'}"] \\ \operatorname{Spec} k \arrow[r] & S' \end{tikzcd}$$

Proposition 1.2.3. *The diagram above defines a family of deformations of X over S'.*

Proof. Both flatness and properness are stable under base change, so we need only prove that the diagram is a cartesian diagram. Notice that, in the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathcal{X} \times_S S' \arrow[r] \arrow[d, "\pi_{S'}"] & \mathcal{X} \arrow[d, "\pi"] \\ \operatorname{Spec} k \arrow[r] & S' \arrow[r] & S \end{tikzcd}$$

the outer square is a pullback, and so is the square on the right. Then, the square on the left is a pullback square because of the pasting property of pullback squares. See https://ncatlab.org/nlab/show/pasting+law+for+pullbacks.

□

We denote the deformation obtained this way by $g_\sharp(S, s, \pi)$.

Definition 1.2.4. *Given a deformation (S, s, π) of X, a morphism $g : S' \to S$ and a closed point $s' \in S'$ mapping to s, we say that the deformation $g_\sharp(S, s, \pi)$ is the pullback of (S, s, π) along g.*

Studying general deformations of a scheme X is in general very hard, so we start by studying the so-called infinitesimal deformations. These are the building blocks for formal deformations, a concept we will introduce in Section 4.2.1.

Definition 1.2.5. *A deformation $\mathcal{X} \to S$ of X is called infinitesimal if S is the spectrum of a ring $R \in \mathcal{A}$. It is called a first order deformation if S is the spectrum of the ring of dual numbers over k.*

When speaking of an infinitesimal deformation $\pi : \mathcal{X} \to S$, we will denote it by referring to the morphism π alone, instead of (S, s, π).

Remark 1.2.1. If $\mathcal{X}$ is an infinitesimal deformation of X, then the inclusion $j : X \to \mathcal{X}$ in the definition of deformation is a homeomorphism. Hence infinitesimal deformations change only the scheme structure of X, and not the topology. In the following, we will identify the topologies on X and on $\mathcal{X}$. If U is an open subset of X we will use the notation $\mathcal{X}|_U$ to denote the subscheme of $\mathcal{X}$ on the open subset U.

Definition 1.2.6. *A deformation $\pi : \mathcal{X} \to S$ is locally trivial if for every $x \in X$ there is an open neighbourhood U of x such that the restriction*

$$\begin{tikzcd} U \arrow[r] \arrow[d] & \mathcal{X}|_U \arrow[d, "\pi"] \\ \operatorname{Spec} k \arrow[r, "s"] & S \end{tikzcd}$$

is a trivial deformation.

Given two morphisms $f : A \to A'$, $g : A' \to A''$ in $\mathcal{A}$, and an infinitesimal deformation η over A, there are natural isomorphisms between $g_\sharp(f_\sharp(\eta))$ and $(gf)_\sharp(\eta)$. Indeed, this follows from functoriality of limits.

Hence, there is a well defined functor of Artin rings $\mathrm{Def}_X : \mathcal{A} \to \mathrm{Set}$, sending each object $A \in \mathcal{A}$ to the set of isomorphism classes of deformations of X over A, and a morphism $f : A \to A'$ to the pullback morphism $g_\sharp$. Functoriality is guaranteed by the remark above.

Remark 1.2.2. Even though the construction of $f_\sharp$ involves a pullback (hence, a controvariance), the deformation functor is a usual functor (that is, not controvariant). This is because, when we take the Spec of a morphism $A' \to A$, the arrows get reversed, and when we take the pullback of a deformation, they get reversed again. This is why, to denote the induced morphism, we used a lower subscript.

Remark 1.2.3. A variant of the functor Def_X that is generally easier to work with is the functor of locally trivial infinitesimal deformations Def'_X. Since locally trivial deformations pull back to locally trivial deformations, there is a well defined assignment

$$\mathcal{A}_\Lambda \ni A \mapsto \mathrm{Def}'_X(A) = \{\text{classes of locally trivial deformations of } X \text{ over Spec } A\}$$

$$(f : A \to A') \mapsto (f_\sharp : \mathrm{Def}'_X(A) \to \mathrm{Def}'_X(A'))$$

that is moreover functorial.

The functors Def_X and Def'_X are used to study the so-called *abstract deformations* of X, that is those that depend solely on the space X. However, for our infinitesimal study of the Hilbert scheme, if we deform a closed subscheme $X \subset \mathbb{P}^n$ we want to remember the subscheme structure of X. So, we need to slightly modify the definition of a deformation, and this leads to the following definition.

Definition 1.2.7. *Let Y a scheme, and $X \subset Y$ a closed subscheme of Y. Then a family of deformations of X in Y is the datum of a commutative diagram*

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d] & \mathcal{X} \arrow[r, "i"] \arrow[d, "\pi"] & Y \times S \arrow[dl, "p"] \\ \operatorname{Spec} k \arrow[r, "s"] & S & \end{tikzcd}$$

with p the projection on the second factor, i a closed immersion and where the square is cartesian. We moreover ask that π be flat surjective, locally of finite presentation and, if S is not a point, proper.

As for abstract deformations, one also calls the fibers the fibers of π deformations of X. The definition of *infinitesimal deformation of X in Y* and *first order deformation of X in Y* are analogous to the abstract case.

The pullback construction done for abstract deformations can be carried out in the same way for embedded deformations. Hence we get a functor of Artin rings, called the *local Hilbert functor*, $H_X^Y : \mathcal{A} \to \mathrm{Set}$. It sends an Artinian ring $A \in \mathcal{A}$ to the set of isomorphism classes of embedded deformations of X over A, and a morphism $f : A \to A'$ to the pullback morphism $f_\sharp$.

Definition 1.2.8. *A closed subscheme $X \subset Y$ is rigid in Y if every embdedded deformation of X in Y is trivial.*

1.3 Moduli problems and the Hilbert scheme

We start by recalling the functorial point of view in algebraic geometry. To do so, we first recall a version of the Yoneda lemma.

Theorem 1.3.1 (Yoneda lemma)**.** *Given a locally small category $\mathcal{C}$, the functor*

$$\mathcal{C} \to \mathrm{Set}^{\mathcal{C}^{op}}$$

$$X \mapsto Hom_{\mathcal{C}}(-, X)$$

is fully faithful.

This means that morphisms between two objects X and Y of $\mathcal{C}$ correspond bijectively to natural transformations between $\mathrm{Hom}_{\mathcal{C}}(-, X)$ and $\mathrm{Hom}_{\mathcal{C}}(-, Y)$. Considering the category of k-schemes (although this is true for any S-scheme), a consequence of the Yoneda lemma is that that, for a scheme X, the scheme is determined up to isomorphism by the functor $\mathrm{Hom}_k(-, X)$, which we call its *functor of points*.

So, we can embed the category of k-schemes into a larger category and view them as functors. The advantages of this point of view is that sometimes, in trying to construct a scheme, it is easier to define its functor of points and then show its existence using general methods. An example of scheme defined in this way is the Hilbert scheme, defined below. Another elementary example showing the philosophy and the methods behind this approach is the construction of the Grassmannian, as done for example in chapter 8 of [WG10].

Moduli problems occupy a central question in algebraic geometry. Roughly speaking, the study of moduli is the study of families of some geometric constructions, like curves, with possibly more strict conditions, like smoothness, of genus g over a field.

The starting point is a family $\mathcal{M}$ over k that we wish to classify up to isomorphism. In this case, by a family we do not mean as in the sense of definition 1.2.1, but a set of geometric objects, like nonsingular connected curves, or line bundles over a fixed variety. Then a family of, say, nonsingular connected curves of genus g over a scheme Y is a flat morphism locally of finite presentation $\mathcal{S} \to Y$ such that the fibers of this morphism are nonsingular connected curves of genus g over k.

Example 1.3.1. A ruled surface S (see example 1.2.3) is a family of genus 0 curves over $\mathbb{P}^1$. To see that it is flat, we use again that S is reduced and that $\mathbb{P}^1$ a Dedekind scheme.

What we would like to do is to find a scheme X "classifying" the family $\mathcal{M}$ (if such a space exists) and a universal family over X such that every other family over every other scheme Y is obtained by pulling back the universal family. By the Yoneda lemma, such a scheme X would define a functor $\mathsf{Sch}\,/k^{\mathrm{op}} \to \mathsf{Set}$. With the functorial point of view in mind, we go the other way around and first define the functor.

Definition 1.3.2. *A moduli problem is a functor $F : \mathsf{Sch}\,/k^{op} \to \mathsf{Set}$.*

Example 1.3.2. In the case of smooth connected curves of fixed genus g, the moduli functor would be

$$F(S) = \{\text{families of nonsingular connected curves of genus } g \text{ over } S\}/\simeq$$

The simplest example of moduli problem is given by corepresentable functor $\mathrm{Hom}_k(-, X)$. I will denote the functor corepresented by a k-scheme X by X, so that $\phi : F \to X$ will mean that ϕ is a natural tranformation from F to $\mathrm{Hom}(-, X)$. To ask that the moduli problem is represented by a scheme is often too much. So, we can weaken the request and have the following definition.

Definition 1.3.3. *A coarse moduli space for a moduli problem F is the datum of a scheme X and of a natural transformation $\phi : F \to X$ inducing an isomorphism $F(k) \simeq X(k)$,*

with the following universal property: for any other scheme Y and pair $(Y, \psi : F \to Y)$ there is a unique morphism of schemes $f : X \to Y$ such that $\psi = f\phi$.
A fine moduli space for a moduli problem F is the datum of a scheme X and of a natural transformation $\phi : F \to X$ such that ϕ is an isomorphism of functors.

If X is a fine moduli for a moduli problem for the family $\mathcal{M}$, then the family $\mathcal{C} \in F(X)$ corresponding to the identity of X via the representing bijection is called the *universal family* of the moduli, and it has the following property. If $\mathcal{S} \to Y$ is a family of objects of $\mathcal{M}$ over a scheme Y, then there is a morphism $f : Y \to X$ such that $\mathcal{S}$ is obtained by pulling back the universal family along f.

Remark 1.3.1. Fine moduli spaces are much rarer to come by than coarse ones. One reason is that, if there is an element of the family $\mathcal{M}$ with nontrivial automorphisms, then there cannot be a fine moduli for $\mathcal{M}$. A qualitative argument for this can be given a follows. If X is a fine moduli and $\mathcal{C}$ the universal family, a family $S \to Y$ yields a morphsim to X as follows: to a point $y \in Y$, we associate the point of X corresponding to the fiber S_y. In particular, if all the fibers are the same scheme T, the induced map $Y \to X$ would be constant. Then, since S would be the pullback of $\mathcal{C} \to X$ along $Y \to X$, we would have that S is the trivial family, that is it is isomorphic to $Y \times T$. However, if T has nontrivial automorphisms we can typically build nontrivial families with all fibers T (the construction is akin to that of a nontrivial bundle: we cover Y with open subsets, on each of these we take the trivial family and then we glue them along nontrivial automorphisms of T).

Example 1.3.3. The projective space $\mathbb{P}^n$ is a fine moduli space for the following functor F. Let T a scheme, and $F(T) = \{(L, x_0, \ldots, x_n)$ such that L is a line bundle on T, and the x_i's are global sections of L that globally generate $L\}$, and for a morphism of schemes $f : T \to S$, then $F(f) : F(S) \to F(T)$ is given by pullback. That $\mathbb{P}$ represents this functor is a consequence of [Har77, II Thm. 7.1].

Example 1.3.4. Consider the moduli problem of smooth genus g curves, see example 1.3.2. If $g \geq 2$, Mumford proved that F has a coarse moduli space $\mathcal{M}_g$ (see [MFK94]), and that its dimension is $3g-3$. Riemann had previously stated that the space of smooth complex curves of genus $g \geq 2$ was $(3g-3)-$ dimensional using a procedure called *counting parameters*. The existence of $\mathcal{M}_g$ shows Riemann's statement for arbitrary algebraically closed fields. On the other hand, using the argument of remark 1.3.1, the existence of ruled surfaces, see remark 1.2.3, implies that there cannot exist a fine moduli space neither of all curves nor of genus 0 curves.

Generally, the existence of a moduli space, be it fine or coarse, is hard to assess. Moreover, even if the existence of such a space was proven, the definition alone usually does not provide much information on the geometric properties of the moduli space, like smoothness or reducedness. Deformation theory can be used to study the local geometry of such spaces, and we will see some examples of how this can be done for a particularly important moduli space, the Hilbert scheme, which we now define.

Let $f : X \to S$ be a projective morphism. For any S-scheme T, let X_T be the scheme $X \times_S T$. The Hilbert functor of X as an S-scheme is the functor $H_{X/S} : \mathrm{Sch}\,/S^{\mathrm{op}} \to \mathrm{Set}$ defined in the following way. To $T \in \mathrm{Sch}\,/S$ we associate the set of closed subschemes of X_T that are flat, proper and locally of finite presentation over T. To a morphism $g : T \to T'$ associate the pullback function, sending $Z \in H_{X/S}(T')$ to the pullback of the following diagram:

$$\begin{tikzcd} & X_T \arrow[d, "g_X"] \\ Z \arrow[r] & X_{T'} \end{tikzcd}$$

Flatness and properness are preserved under pullback. The property of being locally of finite presentation is also stable under base change, so we indeed get an element of $H_{X/S}(T)$, and it is easy to see that this assignement actually assembles to a functor.
The following is a fundamental result of Grothendieck.

Theorem 1.3.4 ([Gro62])**.** *Given a projective morphism $X \to \operatorname{Spec} k$, the Hilbert functor $H_{X/k}$ has a fine moduli space that is morevoer a disjoint union of projective schemes*

We denote the scheme in Grothendieck's theorem by Hilb_X, and we call it the Hilbert scheme of X.

Remark 1.3.2. It follows from the definitions that elements of $\mathrm{Hilb}_X(k)$ correspond bijectively to closed subschemes of X. Hence, since k is algebraically closed, closed subschemes of X correspond to closed points of Hilb_X.

We can modify the definition of Hilbert scheme as to only consider closed subschemes with fixed Hilbert polynomial. This is useful if we want to study for example the family of curves in some projective scheme, or the family of length n subschemes of a scheme.
Let $X \subset \mathbb{P}^n$ a projective scheme.

Definition 1.3.5. *For a coherent sheaf $\mathcal{F}$ on X, the function $p_{\mathcal{F}} : \mathbb{N} \to \mathbb{N}$ defined by $p_{\mathcal{F}}(n) = \sum_{i=0}^{n}(-1)^i h^i(X, \mathcal{F}(n))$ is called the Hilbert polynomial of $\mathcal{F}$.*

Remark 1.3.3. The function in the previous definition is indeed a polynomial function od degree equal to the dimension of the support of $\mathcal{F}$, see [Har77, III Ex. 5.2].

The hilbert polynomial of the structure sheaf of X is also called the Hibert polynomial of X, and this will be the meaning we will use.

With the Hilbert polynomial in hand, we can define the variant of the Hilbert functor.
Let P a numerical polynomial. Consider the functor H_X^P sending a scheme T to the set of closed subschemes Z of X_T that are flat, proper and locally of finite presentation over T and such that $Z(t)$ has Hilbert polynomial P for all $t \in T$. A result similar to 1.3.4 holds for these functors as well, namely that H_X^P are representable by a scheme Hilb_X^P.

Example 1.3.5. The Hilbert scheme of plane curves is $\mathrm{Hilb}_{\mathbb{P}^2}^{\mathrm{curves}} = \bigsqcup_P \mathrm{Hilb}_{\mathbb{P}^2}^P$ where the union is taken among all numerical polynomials of degree 1. We will show in chapter 3 that it is smooth.

Chapter 2

Tangent Spaces

We discuss how to define in a natural way a notion of tangent space to a functor of Artin rings satisying some mild conditions, and then compute the tangent space to the local Hilbert functor. In the third section, we show how this computation can be applied to studying the tangent space of the Hilbert scheme at a closed point.

Throughout the section, we work with a fixed closed immersion $X \subset Y$ of algebraic schemes.

2.1 Tangent spaces of functors

We saw in 1.1.1 that, given a functor of Artin rings F and a diagram

$$\begin{tikzcd} A' \arrow[dr] & & A'' \arrow[dl] \\ & A & \end{tikzcd}$$

in $\mathcal{A}_\Lambda$, there was a natural map $\alpha : F(A' \times_A A'') \to F(A') \times_{F(A)} F(A'')$. Schlessinger introduced in his paper [Sch68] the following conditions:

- $H_0)$: $F(k)$ is a singleton;
- $H_1)$: if $A' = k[\varepsilon]$, $A = k$ then α is bijective;
- $H_2)$: if $A' \to A$ is a small extension, then α is surjective;
- $H_3)$: if $A' = A''$, $\phi = \psi$ and ϕ is a small extension then α is bijective.

Their interest lies in the fact that the existence of miniversal and universal formal families for a functor of Artin rings can be deduced by verifying that some of the conditions above are satisfied. We will talk about these concepts in chapter 4.

To begin, consider a functor of Artin rings $F : \mathcal{A}_\Lambda \to \text{Set}$ satisfying conditions $H_0)$ and $H_1)$.

Proposition 2.1.1. *If F is as above, then $F(k[\varepsilon])$ has a structure of of k-vector space, natural in the sense that if G satisfies $H_0)$ and $H_1)$ and $\phi : F \to G$ is a morphism of functors of Artin rings, then $\phi_{k[\varepsilon]}$ is k-linear.*

Proof. Notice that the two conditions imply the chain of isomorphisms

$$F(k[\varepsilon] \times_k k[\varepsilon]) \simeq F(k[\varepsilon]) \times_{F(k)} F(k[\varepsilon]) \simeq F(k[\varepsilon]) \times F(k[\varepsilon])$$

where we first use $H_1)$ and then $H_0)$.
We define the sum on $F(k[\varepsilon])$ as follows: consider the morphism $+ : k[\varepsilon] \times_k k[\varepsilon] \to k[\varepsilon]$ defined by $(a + b\varepsilon, a + d\varepsilon) \mapsto a + (b + d)\varepsilon$. Then the sum on $F(k[\varepsilon])$ is given by the composition

$$F(+) \cdot \alpha^{-1} : F(k[\varepsilon]) \times F(k[\varepsilon]) \to F(k[\varepsilon] \times_k k[\varepsilon]) \to F(k[\varepsilon])$$

We now define multiplication by $t \in k$ as follows: consider the morphism $t\cdot : k[\varepsilon] \to k[\varepsilon]$ defined by $a + b\varepsilon \mapsto a + (tb)\varepsilon$. We need to show that the operations so defined indeed yield a structure of k-vector space on $F(k[\varepsilon])$. We verify here only that they define an abelian group structure, for the other verifications are done similarly.
Let $(+, id)$ (resp. $(id, +)$) the morphism from $k[\varepsilon] \times_k k[\varepsilon] \times_k k[\varepsilon]$ to $k[\varepsilon] \times_k k[\varepsilon]$ which is $+$ on the first two copies of $k[\varepsilon]$ and the identity on the third (the identity on the first and the sum on the last two copies). Since

$$\begin{tikzcd} k[\varepsilon] \times_k k[\varepsilon] \times_k k[\varepsilon] \arrow[r, "(id,+)"] \arrow[d, "(+,id)"] & k[\varepsilon] \times_k k[\varepsilon] \arrow[d, "+"] \\ k[\varepsilon] \times_k k[\varepsilon] \arrow[r, "+"] & k[\varepsilon] \end{tikzcd}$$

commutes, once we apply F we see that the sum on $F(k[\varepsilon])$ is associative. Let inv it the endomorphism of $k[\varepsilon] \times_k k[\varepsilon]$ that acts as follows: $inv(a + b\varepsilon, a + c\varepsilon)) = (a + c\varepsilon, a + b\varepsilon)$. Then inv commutes with $+$, so applying F again we see that the sum is commutative.
Let $i : k \to k[\varepsilon]$ be the inclusion, and e the composition $k[\varepsilon] \to k \to k[\varepsilon]$. Then, the diagram

$$\begin{tikzcd} k[\varepsilon] \arrow[r, "e\times id"] \arrow[dr] & k[\varepsilon] \times_k k[\varepsilon] \arrow[d, "m"] & k[\varepsilon] \arrow[l, "id\times e"'] \arrow[dl] \\ & k & \end{tikzcd}$$

commutes. Since $F(k)$ is a singleton, we see that the image of $F(e)$ is the unit of $F(k[\varepsilon])$. Finally, to see that there is an additive inverse, notice that there is a commutative diagram

$$\begin{tikzcd} k[\varepsilon] \arrow[r, "\tau\times id"] \arrow[dr] & k[\varepsilon]\times_k k[\varepsilon] \arrow[d, "+"] & k[\varepsilon] \arrow[l, "id\times\tau"'] \arrow[dl] \\ & k & \end{tikzcd}$$

where τ sends $a+b\varepsilon$ to $a-b\varepsilon$. Applying F to this diagram, we conclude. Considering now the leg of ϕ at $k[\varepsilon]$, the verification that $\phi_{k[\varepsilon]}$ is k-linear is done again by standard computations similar to the one above. □

In view of the previous proposition, we give the following definition:

Definition 2.1.2. *Consider a functor of Artin rings F satisfying conditions H_0) and H_1). We then call the k-vector space $F(k[\varepsilon])$ the tangent space of F, and we denote it by t_F. If G satisfies the same conditions, the differential of ϕ is the linear map $\phi_{k[\varepsilon]} : t_F \to t_G$ which we denote by $d\phi$.*

As the geometric terminology suggests, the tangent space to a functor of Arting rings and the differential of a morphism have indeed a geometric meaning. We will see how in section 2.3.

For functors F satisfying condtions H_0) and H_1) we can then add the following condition:

- H_4): t_F is finite-dimensional as a k-vector space.

2.2 First order deformations

In the previous section we defined the tangent space to a functor of Artin rings. We will now compute the tangent spaceto the functor H_X^Y. We will need the following result, often called the local criterion for flatness:

Theorem 2.2.1 ([Mat86, Theorem 22.3])**.** *Consider a surjection $A \to A'$ in $\mathcal{A}_\Lambda$. Let M an A-module. Then M is flat if and only if $M \otimes_A A'$ is flat as an A' module and $\mathrm{Tor}_1^A(M, A') = 0$.*

An application of the local criterion is the following proposition, which will be crucial in proving that the local Hilbert functor satisfies the Schlessinger conditions.

Consider the following situation: we are given maps of Artin rings $A' \to A$ and $A'' \to A$, with the latter surjective. Suppose we are also given modules M', M'', M over A', A'', A respectively, with maps $M' \to M$, $M'' \to M$ compatible with the module structures. Suppose moreover that the maps $M' \otimes_{A'} A \to M$ and $M'' \otimes_{A''} A \to M''$ are isomorphisms. Finally, we assume that M' is flat over A' and M'' is flat over A''. Denote $A' \times_A A''$ and $M' \times_M M''$ by $\bar{A}$ and $\overline{M}$.

Lemma 2.2.2. *Keeping the above notations, there are isomorphisms $\overline{M} \otimes_{\bar{A}} A'' \to M''$ and $\overline{M} \otimes_{\bar{A}} A' \to M'$. Moreover, $\overline{M}$ is flat over $\bar{A}$.*

Proof. Consider the following diagram

$$
\begin{tikzcd}
0 \arrow[r] & \ker f \arrow[r] \arrow[d] & \overline{M} \arrow[r, "f"] \arrow[d] & M' \arrow[d] \\
0 \arrow[r] & \ker g \arrow[r] & M'' \arrow[r, "g"] & M
\end{tikzcd}
$$

It commutes by definition of $\overline{M}$. Since $A'' \to A \to 0$ is exact, applying $- \otimes_{A''} M''$, we find that g is surjective. Diagram chasing now shows that f is also surjective. If we now tensor the upper row by A', we find that $\overline{M} \otimes_{\bar{A}} A' \to M'$ is surjective. Similarly, one sees that $\bar{A} \to A$ is surjective, with kernel $(0, a'')$ where a'' is in the

kernel of $A'' \to A$. We prove injectivity: if $\sum(m_i', m_i'') \otimes b_i$ goes to zero, then $\sum a_i' \cdot m_i' = 0$, where a_i' (resp. a_i'') is the image of $b_i \in \bar{A}$ in A' (in A''). But $\sum(m_i', m_i'') \otimes b_i = \sum(a_i' \cdot m_i', a_i'' \cdot m_i'') \otimes 1 = (0, \sum a_i'' \cdot m_i'') \otimes 1$ and we deduce that the image of $\sum a_i'' \cdot m_i''$ in M is 0. So, when tensoring by $- \otimes_{\bar{A}} A'$, the element $\sum a_i'' \cdot m_i''$ goes to zero, and hence $\overline{M} \otimes_{\bar{A}} A' \to M'$ is indeed an isomorphism of A'-modules.
If I is the kernel of $A'' \to A$, tensoring the sequence $0 \to I \to A'' \to A \to 0$ with $- \otimes_{A''} M''$ to get

$$0 \longrightarrow I \otimes_{A''} M'' \longrightarrow M'' \longrightarrow M \longrightarrow 0$$

This sequence is exact beacuse of flatness of M''. Moreover, there is an isomorphism

$$I \otimes_{A''} M'' \simeq I \otimes_A (A \otimes_{A''} M'') \simeq I \otimes_A M$$

Going back to the diagram above, we found that $\ker g$ is $I \otimes_A M$. Moreover, diagram chasing shows that there is a bijection between $\ker f$ and $\ker g$. Consider now the following morphism: $\bar{A} \to A'$, $(a', a'') \mapsto a'$. Since $A'' \to A$ is surjective, this morphism is surjective, too. We can compute its kernel: it's made of the pairs $(0, a'')$ such that a'' goes to 0 under the projection $A'' \to A$, so it is isomorphic to I.
The following chain of isomorphism shows then that $\ker f$ is isomorphic to $I \otimes_{\bar{A}} \overline{M}$.

$$\ker f \simeq I \otimes_A M \simeq I \otimes_A (A \otimes_{A'} M') \simeq I \otimes_{A'} M' \simeq I \otimes_{A'} (A' \otimes_{\bar{A}} \overline{M}) \simeq I \otimes_{\bar{A}} \overline{M}$$

So, the sequence $0 \to I \otimes_{\bar{A}} \overline{M} \to \overline{M} \to M' \to 0$ is exact. We conclude that

$$\mathrm{Tor}_1^{\bar{A}}(\overline{M}, A') = 0$$

Moreover, we proved above that $\overline{M} \otimes_{\bar{A}} A' \simeq M'$, so it is flat over A'. We conclude by theorem 2.2.1 that $\overline{M}$ is flat over $\bar{A}$.
It only remains to prove that he map $\overline{M} \otimes_{\bar{A}} A'' \to M''$ is an isomorphism. We tensor the sequence $0 \to I \to A'' \to A \to 0$ with $- \otimes_{\bar{A}} \overline{M}$, to get the sequence

$$0 \to I \otimes_{\bar{A}} \overline{M} \to A'' \otimes_{\bar{A}} \overline{M} \to A \otimes_{\bar{A}} \overline{M} \to 0$$

The first non-zero term is isomorphic to $I \otimes_{A''} M''$ (we proved it above when juggling with the kernels), and the last non-zero term is

$$A \otimes_{\bar{A}} \overline{M} \simeq A \otimes_{A'} (A' \otimes_{\bar{A}} \overline{M}) \simeq A \otimes_{A'} M' \simeq M$$

where the last isomorphism follows from our initial assumptions. If we then compare the sequence we obtained with $0 \to I \otimes_{A''} M'' \to M'' \to M \to 0$, the 5-lemma implies the desired isomorphism. $\square$

If X is a topological space, $\mathcal{F}$, $\mathcal{S}$ and $\mathcal{O}$ are sheaves of rings over X together with morphisms of sheaves of rings $\mathcal{F} \to \mathcal{O}$, $\mathcal{S} \to \mathcal{O}$, then we can define another sheaf $\mathcal{F} \times_{\mathcal{O}} \mathcal{S}$ on X by $(\mathcal{F} \times_{\mathcal{O}} \mathcal{S})(U) = \mathcal{F}(U) \times_{\mathcal{O}(U)} \mathcal{S}(U)$ for every open subset U of X. It is the fibered product of $\mathcal{F}$ and $\mathcal{S}$ over $\mathcal{O}$ in the category of sheaves over X.

Theorem 2.2.3. *The functor H_X^Y satisfies the Schlessinger's conditions H_0) through H_3).*

Proof. Clearly $H_X^Y(k) = \{X \to Y\}$, so H_0) is verified.
For the other conditions, we prove something stronger, that is that, for every diagram

$$A' \xrightarrow{\psi} A \xleftarrow{\phi} A''$$

where ϕ is a small extension, then the map α is a bijection. This will imply all the Schlessinger conditions.
Call $\bar{A} = A' \times_A A''$. Consider an element of $H^Y_X(A') \times_{H^Y_X(A)} H^Y_X(A')$, that is a pair of embedded deformations $\xi' \in H^Y_X(A')$, $\xi'' \in H^Y_X(A'')$ such that both ξ' and ξ'' pullback to the same deformation ξ over A

$$\xi' \times_{\operatorname{Spec} A'} \operatorname{Spec} A \simeq \xi \simeq \xi'' \times_{\operatorname{Spec} A''} \operatorname{Spec} A$$

Let $\mathcal{O}'$, $\mathcal{O}''$ and $\mathcal{O}$ the structure sheaves of ξ', ξ'' and ξ, respectively. These are all sheaves on the same topological space X by remark 1.2.1. Consider the locally ringed space $\bar{\xi} = (X, \mathcal{O}' \times_O \mathcal{O}'')$. We can work locally to see that $\bar{\xi}$ is a scheme. There is a morphism $\xi' \to \bar{\xi}$ which is the identity on the underlying topological space, and the sheaf map is $id_*(\mathcal{O}' \times_{\mathcal{O}} \mathcal{O}'') = \mathcal{O}' \times_{\mathcal{O}} \mathcal{O}'' \to \mathcal{O}'$ is the structure map of the pullback, and a similar morphism $\xi'' \to \bar{\xi}$. From the properties of the fibered product, it follows that these morphisms make the following diagram a pushout diagram

$$\begin{tikzcd}
\xi \arrow[r] \arrow[d] & \xi' \arrow[d] \\
\xi'' \arrow[r] & \bar{\xi}
\end{tikzcd}$$

From the definition of $\bar{\xi}$, it also follows readily that it is a scheme over $\operatorname{Spec} \bar{A}$: one checks it locally using that both ξ' and ξ'' are schemes. Since flatness can be checked locally, we can apply lemma 2.2.2 to conlcude that it is flat an $\bar{A}$-scheme.

The following diagram, where all the maps are the "natural" ones, is commutative

$$\begin{array}{ccc} \xi' & \longrightarrow & \operatorname{Spec} A' \\ \downarrow & & \downarrow \\ \bar{\xi} & \longrightarrow & \operatorname{Spec} \bar{A} \end{array}$$

so there is an induced morphism $\xi' \to \bar{\xi} \times_{\operatorname{Spec} \bar{A}} \operatorname{Spec} A'$. We can use again lemma 2.2.2 to check locally that this is an isomorphism, that is, $\bar{\xi}$ pulls back to ξ'. In the same way one proves that $\bar{\xi}$ pulls back to ξ'' along $\operatorname{Spec} A'' \to \operatorname{Spec} \bar{A}$.
We now have to prove that $\bar{\xi}$ is an element of $H_X^Y(\bar{A})$. The compositions

$$\xi' \to \operatorname{Spec} A' \times Y \to \operatorname{Spec} \bar{A} \times Y$$

$$\xi'' \to \operatorname{Spec} A'' \times Y \to \operatorname{Spec} \bar{A} \times Y$$

restrict to the same morphism on ξ, and the first one is a closed emebdding, because composition of closed embeddings. Again by the pushout property we get a morphism $\phi : \bar{\xi} \to \operatorname{Spec} \bar{A} \times Y$. Moreover, since $\xi' \to \bar{\xi}$ is a closed embedding with square-zero kernel sheaf and $\xi' \to \operatorname{Spec} \bar{A} \times Y$ is a closed embedding, ϕ is a closed embedding, too.
Finally, since both ξ' and ξ'' are locally of finite presentation over A' and A'',$\bar{\xi}$ is locally of finite presentation over $\bar{A}$. To prove this we work locally, so that we restrict to proving a proposition similar to lemma 2.2.2, but with the property of being of finite presentation rather than being flat, and it is immediate to see that this is true. So we have finally proven that $\bar{\xi}$ in an element of $H_X^Y(\bar{A})$. Moreover, it maps to (ξ', ξ'') under α, so α is surjective.
If $\bar{\xi}'$ is another preimage of (ξ', ξ''), then the pushout property yields a morphism

$\bar{\bar{\xi}} \to \bar{\bar{\xi}}'$ which is also an isomorphism. Since the following diagram commutes

$$\begin{tikzcd} & Y \times \operatorname{Spec} \bar{A} & \\ \bar{\bar{\xi}} \arrow[ur] \arrow[rr] & & \bar{\bar{\xi}}' \arrow[ul] \end{tikzcd}$$

we conclude that $\bar{\bar{\xi}} = \bar{\bar{\xi}}'$, so α is also injective, which concludes the proof. □

Since H_X^Y satisfies $H_0)$ and $H_1)$, we saw in the previous section that the first order emebdded deformations of X in Y form a k-vector space. Our next objective is to compute this space. Properties $H_2)$ and $H_3)$ will be relevant in the next two chapters.
We will need the following well known homological lemma:

Lemma 2.2.4 (9-lemma)**.** *Suppose we are given in a diagram*

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & A' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & B' \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & B'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & C' \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & C'' \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

in an abelian category, where all the squares are commutative and the columns are exact. Then

- *If the bottom two rows are exact, so is the top row*
- *If the top two rows are exact, so is the bottom row.*

We can now prove the main result of this section. First, we need to recall a definition.

Definition 2.2.5. *Let $X \to Y$ a closed embedding corresponding to a sheaf of ideals $\mathcal{I}$. The normal sheaf of X in Y is the sheaf $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_X)$. We will use the notation $\mathcal{N}_{X/Y}$ to denote it.*

Theorem 2.2.6. *Let $X \to Y$ a closed embedding. Then the tangent space to H_X^Y is $H^0(X, \mathcal{N}_{X/Y})$.*

Proof. Step 1: affine case In this case, $Y = \operatorname{Spec} B$ for some k-algebra B, and X is $\operatorname{Spec} B/I$. Rewriting the definition of first order deformation in the language of affine scheme, we see that first order embedded deformations of X correspond to ideals $I' \subset B' = B[t]/(t^2) = k[\varepsilon] \otimes B$ such that $k[\varepsilon] \to B'/I'$ is a flat morphism and $B'/I' \otimes k = B/I$. Moreover, in the affine case, $H^0(X, \mathcal{N}_{X/Y}) = \operatorname{Hom}_{B/I}(I/I^2, B/I) \simeq \operatorname{Hom}_B(I, B/I)$.

We start by, given an embedded deformation corresponding to the ideal I', constructing an element of $\mathrm{Hom}_B(I, B/I)$. Consider the following diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & I \arrow[r, "t\cdot"] \arrow[d] & I' \arrow[r] \arrow[d] & I \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B \arrow[r, "t\cdot"] \arrow[d] & B' \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B/I \arrow[r, "t\cdot"] \arrow[d] & B'/I' \arrow[r] \arrow[d] & B/I \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The middle row is exact, so by lemma 2.2.4 exactness of either the first or the last row implies exactness of the other one.
Since we are given an element of $H_X^Y(k[\varepsilon])$, the bottom row of the diagram is exact: indeed, it is obtained by taking the tensor product of the sort exact sequence

$$0 \to k \to k[\varepsilon] \to k \to 0$$

with the module B'/I', and the resulting short sequence is exact since B'/I' is flat. Hence, the first row is exact, too. Given $x \in I$, let $x + ty \in I'$, $x + ty' \in I'$ be two liftings of x. Then $t(y - y') \in tI$ by exactness of the first row, so $\bar{y} \in B/I$ is well defined. It can now easily be shown that the assignement $x \mapsto \bar{y}$ yields a B-module homomorphism $I \to B/I$.
We now do the opposite: given an element of $\mathrm{Hom}_B(I, B/I)$, we build a first order embedded deformation. Suppose then we are given a B-morphism $\phi : I \to B/I$.

Let $\pi : B \to B/I$ denote the projection. The set $I' = \{x + ty,\ y \in B$ such that $x \in I$ and $\phi(x) = \pi(y)\}$ is an ideal of $B[t]/(t^2)$. It is clear that $B'/I' \otimes k \simeq B/I$, so we only need to check that B'/I' is flat as a $k[\varepsilon]$-algebra. We have that the image of I' under the projection $B' \to B$ is exactly I, and the short sequence

$$0 \to I \xrightarrow{t\cdot} I' \to I \to 0$$

is exact. So, using again 2.2.4 we get that the bottom row of the diagram above is exact. This implies that $\mathrm{Tor}_1^{k[\varepsilon]}(B'/I', k) = 0$, so we conclude by applying the local criterion for flatness.

The two constructions are mutual inverses, so we obtain the desired isomorphism between $H_X^Y(k[\varepsilon])$ and $\mathrm{Hom}_B(I, B/I)$.

Step 2: global case Let $\mathcal{I}$ be the ideal sheaf of $i : X \to Y$. We cover Y with affine sets. The construction above is compatible with localizations, so we can glue the isomorphism locally to get a one-to-one correspondence between elements of $H_X^Y(k[\varepsilon])$ and elements of $H^0(\mathscr{H}om_{\mathcal{O}_Y}(\mathcal{I}, i_*\mathcal{O}_X)) \simeq H^0(\mathscr{H}om_{\mathcal{O}_X}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_X)) = H^0(X, \mathcal{N}_{X/Y})$. □

Corollary 2.2.7. *If X is proper, then H_X^Y satisfies all the Schlessinger conditions H_0) through H_4).*

Proof. The first 4 conditions are satisfied by theorem 2.2.3. Since the tangent space is given by the global sections a coherent sheaf on X, condition H_4) follows from properness. □

In the case that interests us, that is in the study of the Hilbert scheme, Y is a projective scheme, and so X is too, so that the corollary will be satisfied.

2.3 Tangent space of the Hilbert scheme

We assume Y to be projective over k. As with all deformation theory results, the computations of the previous section have a geometric interpretation for moduli problems. We recall the definition of tangent space for k-rational points:

Definition 2.3.1. *Let X a scheme and $x \in X(k)$. Let $\mathfrak{m}_x$ the maximal ideal of the local ring of x. The tangent space of X at x is the k-vector space $T_{X,\,x} = (\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee$.*

It is easy to see that there is the following equivalent definition: if $A \in \mathcal{A}$, let $X(A)_x$ denote the set of morphisms $\operatorname{Spec} A \to X$ having image x. This defines a functor of Artin rings $X(-)_x : \mathcal{A} \to \mathrm{Set}$.

Proposition 2.3.2. *The tangent space of $X(-)_x$ is isomorphic to $T_{X,\,x}$.*

Proof. A morphism $\operatorname{Spec}(k[\varepsilon]) \to X$ with image x is the datum of a local morphism of k-algebras $\phi : \mathcal{O}_{X,x} \to k[\varepsilon]$, so we prove that the space of the latter ones is isomorphic to $(\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee$. Since ϕ is local, $\phi(\mathfrak{m}_x) \subset \varepsilon k \simeq k$, and moreover since the latter is square-zero, $\phi(\mathfrak{m}_x^2) = 0$, so we get a well defined morphism $\bar{\phi} : \mathfrak{m}_x/\mathfrak{m}_x^2 \to k$. Conversely, given such a $\bar{\phi}$, let $\phi : \mathfrak{m}_x \to \varepsilon k$ defined by $m \mapsto \varepsilon\phi(m \mod \mathfrak{m}_x^2)$. Now, every element of $\mathcal{O}_{X,x}$ can be written in a unique way as the sum of an element of k and an element of $\mathfrak{m}_x$, so if $a = b + m$, with $b \in k$ and $t \in \mathfrak{m}_x$, we define $\phi(a)$ to be equal to $a + \phi(t)$. It can now be checked that this is a local morphism of k-algebras and that the two constructions are inverses to each other. □

This shows that the notion of tangent space to a functor of Artin rings is really of geometric nature. This can also be said for the differential (see definition 2.1.2). Indeed, given a morphism $f : X \to Y$, we recall here the definition of differential of f. If $x \in X(k)$ maps to $y \in Y(k)$ (notice that if x is k-rational, its image must be k-rational as well), then the local map $\phi : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ induces a morphism

$\bar{\phi} : \mathfrak{m}_y/\mathfrak{m}_y^2 \to \mathfrak{m}_x/\mathfrak{m}_x^2$, which becomes a map $df : T_{X,x} \to T_{Y,y}$ by taking the dual. We call this map the differential of f at x.
Notice that, in this situation, there is also a morphism of functors of Artin rings $f_* : X(-)_x \to Y(-)_y$, obtained by post-composing with ϕ. Then, with computations as above, we can show that the "functor theoretic" differential of f_* is indeed the "geometric" differential df.

With the discussion above, we can now find some first informations on Hilbert schemes. Indeed, all of this chapter's work gives us the necessary tools to finding the tangent space to points in the Hilbert scheme.
Let $X \subset Y$ be a closed subscheme and x the corresponding closed point of Hilb_Y (see remark 1.3.2). Unwinding the definition of the Hilbert scheme, we see that elements of $\mathrm{Hilb}_Y(A)_x$ correspond bijectively with proper flat subschemes of $Y \times A$ that pullback to X in $Y \times \operatorname{Spec} k = Y$, that is to elements of $H_X^Y(A)$. The bijection is natural, hence we get the following result.

Proposition 2.3.3. *Let $X \subset Y$ a closed subscheme and x the corresponding closed point of Hilb_Y. Then there is an isomorphism of functors $H_X^Y \simeq \mathrm{Hilb}_Y(-)_x$.*

Evaluating the two functors in $k[\varepsilon]$, we get the following description for the tangent space of the Hilbert scheme.

Corollary 2.3.4. *The tangent space of Hilb_Y at x is $H^0(X, \mathcal{N}_{X/Y})$.*

Remark 2.3.1. The arguments above imply the prorepresentability of H_X^Y. Indeed, X corresponds to a closed point x of Hilb_Y. Using proposition 2.3.3 immediate to see that H_X^Y is prorepresented by $\widehat{\mathcal{O}_{\mathrm{Hilb}_Y, x}}$. A proof of this fact not relying on the difficult result that Hilb_Y is representable can be given using Schlessinger's criterions.

Remark 2.3.2. In the same way as above one can deduce what the tangent spaces to fine moduli spaces $\mathcal{M}$. One considers the functor F of infinitesimal deformations of an object X of the family $\mathcal{M}$ and computes its tangent space. Arguments similar to the ones above show that the tangent space to the moduli space in the point corresponding to X is the t_F. See [Ser06, Chapter 1.2] and [Ser06, Chapter 2.4] for a study of the functors Def_X and Def'_X and their tangent space. For deformations of line bundles, and more in general, quasicoherent sheaves, see [Ser06, Chapter 3.3], [Har10, Chapter 6] and [Har10, Chapter 7].

Chapter 3

Liftings and obstructions

In this section we fix a functor of Artin rings $F : \mathcal{A}_\Lambda \to \mathrm{Set}$. All extensions will be Λ-extension, and so we will drop the prefix Λ. As in the previous section, we work with a fixed closed immersion $X \subset Y$ of algebraic schemes.

In the first part, given a small extension of artin rings, we define what are the lifings of a fiexed element $\xi \in F(A)$, and show how these liftings interact with the tangent space t_F of F. We also introduce the concept of *obstruction* to a lifting and obstruction space. In the second, as usual, we specialize to the local Hilbert functor. In the third part we apply all this machinery to proving the smoothness of some Hilbert schemes.

3.1 Liftings

Definition 3.1.1. *Given a small extension $\phi : A' \to A$ and an element $\xi \in F(A)$, a lifting of ξ to A' is an element $\tilde{\xi} \in F(A')$ such that $F(\phi)(\tilde{\xi}) = \xi$. We denote by* $\mathrm{Lif}(A, A', \xi)$ *the set of liftings of $\xi \in F(A)$ to A'.*

We now show that, if F satisfies all the Schlessinger conditions, there is a close relationship between the tangent space to F and the set of liftings.

Theorem 3.1.2. *Suppose the functor F satisifes all the Schlessinger conditions H_0) through H_4). Let $\phi : A' \to A$ a small extension, and $\xi \in F(A)$. Then there is a free and transitive action of t_F on $\mathrm{Lif}(A, A', \xi)$.*

To prove theorem we start by proving smaller pieces.

Proposition 3.1.3. *If F satisfies conditions H_0) and H_1) there is an isomorphism θ : $F(A' \times_A A') \to F(A') \times t_F$.*

Proof. We start by considering the following map: $\beta : A' \times_A A' \to A' \times_k k$ defined by $(a, b) \mapsto (a, \bar{a} + \varepsilon(a - b))$. Here $\bar{a}$ is the image of a under the quotient $A \to k$, and the difference $a - b$ is an element of k since it is an element of the kernel of ϕ, and by remark 1.1.3 the kernel is isomorphic to k. Then we show that β is a map of Λ-algebras and that it is an isomorphism.
It is clearly additive, so we verify that it preserves products. Given also $(a', b') \in A' \times_A A'$, we have

$$\beta(aa', bb') = (ab, \bar{ab} + \varepsilon(aa' - bb')$$

while

$$\beta(a, b) \cdot \beta(a', b') = (aa', \bar{a}(\bar{b})) + \bar{a}\varepsilon(a' - b') + \bar{a}'(a - b)$$

Write $\bar{a} = a + t$, with $t \in \ker(\phi)$. Since $a - b \in \ker(\phi)$ and $\ker(\phi)^2 = 0$, then

$$\bar{a}\varepsilon(a' - b') = \varepsilon a(a' - b')$$

So, the ε-component of $\beta(a, b) \cdot \beta(a', b')$ is $aa' - ab' + a'a + a'b$. But,

$$-ab' + aa' + a'b - bb' = a(a' - b') - b(a' - b') = (a - b)(a' - b') = 0$$

So, $aa' - ab' + a'a + a'b = aa' - bb'$ and hence β is a ring morphism. Moreover, since Λ acts on both rings by multiplication on each coordinate, so the morphism

is clearly of Λ-algebras.
If $\beta(a,b)$ goes to zero, then a must be zero, so b must be zero too. Conversely, given $(a,\bar{a}+\varepsilon t) \in A' \times_k k[\varepsilon]$, then $(a, a-t)$ is a preimage of said element. Here we see t as an element of A' via $t \in k = \ker(\phi) \subset A'$. We conclude that β is an isomorphism.
Then, the composition

$$F(A' \times_A A') \to F(A') \times_{F(k)} F(k[\varepsilon]) \to F(A') \times t_F$$

yields the desired isomorphism. Here the first map is the composition of $F(\beta)$ and the map α of remark 1.1.1, and it is an isomorphism because $F(\beta)$ is and because α is a bijection because of property H_1). The second map is an isomorphism because of property H_0). □

The isomorphism θ has the following property: if π_1 is the projection $A' \times_A A' \to A'$ on the first coordinate, and if $\tilde{\pi} : F(A') \times t_F \to F(A')$ is also the projection on the first coordinate, then the following diagram commutes

$$\begin{tikzcd} F(A' \times_A A') \arrow[r, "\theta"] \arrow[d, "F(\pi_1)"] & F(A') \times t_F \arrow[d, "\tilde{\pi}"] \\ F(A') \arrow[r, "id"] & F(A') \end{tikzcd}$$

Consider now the following map:

$$\mu : F(A') \times t_F \to F(A' \times_A A') \to F(A')$$

where the second map is the projection on the second coordinate.

Proposition 3.1.4. *The composition above defines an action of t_F on $F(A')$.*

Proof. We start by verifying that $\mu(\xi, 0) = \xi$ for all $\xi \in F(A')$. Let $\delta : A' \to A' \times_A A'$ be the diagonal map $a \mapsto (a, a)$. Under the bijection $F(A') \times t_F \simeq F(A' \times_A A')$, elements of the form $(\xi, 0)$ correspond to the image of $F(\delta)$

$$\begin{tikzcd}
F(A') \times t_F \arrow[dr, "\simeq"] \arrow[ddrr, bend left, "\mu"] & & \\
F(A') \arrow[u, "\iota"] \arrow[r, "f(\delta)"] & F(A' \times_A A') \arrow[d, "\alpha"] \arrow[dr, "F(\pi)"] & \\
& F(A') \times_{F(A)} F(A') \arrow[r] & F(A')
\end{tikzcd}$$

where ι is the map $x \mapsto (x, 0)$ and π the projection on the second factor.
Following the above diagram one finds that indeed $\mu(\xi, 0) = \xi$.
We now verify the additivity of the action, that is that for every $\xi \in F(A')$ and $a, b \in t_F$ we have $\mu(\mu(\xi, a), b) = \mu(\xi, a + b)$. First, we notice that we can show that there is an isomorphism between $A' \times_A A' \times_A A'$ and $A' \times_k k[\varepsilon] \times_k k[\varepsilon]$ as done in proposition 3.1.3, and we get an bijection $F(A' \times_A A' \times_A A') \simeq F(A') \times t_F \times t_F$. We need to show that the diagram

$$\begin{tikzcd}
F(A') \times t_F \times t_F \arrow[r, "\mu \times id"] \arrow[d, "id \times F(+)"] & F(A') \times t_F \arrow[d, "\mu"] \\
F(A') \times t_F \arrow[r, "\mu"'] & F(A')
\end{tikzcd}$$

commutes. But following the right-then-down direction of the diagram amounts to applying F to the projection on the third factor $A' \times_k k[\varepsilon] \times_k k[\varepsilon] \simeq A' \times_A A' \times_A A' \to A'$. On the other hand, the map $id \times F(+)$ is induced by the morphism $A' \times_k k[\varepsilon] \times_k k[\varepsilon] \to A' \times_k k[\varepsilon]$ induced by $(a, \bar{a} + b\varepsilon, \bar{a} + c\varepsilon) \mapsto (a, \bar{a} + (b + c)\varepsilon)$.

Then, to get μ we apply F to the projection on the second factor $A' \times_k k[\varepsilon] \simeq A' \times_A A' \to A'$. Since we can verify that both the compositions are the same as maps of rings, we get that the diagram above commutes. $\square$

We can now prove theorem 3.1.2: we have a map

$$F(A') \times t_F \simeq F(A' \times_A A') \to F(A') \times_{F(A)} F(A')$$

Since we assume that F satisfied all the Schlessinger conditions, the map above is a bijection. Moreover, we can explicitly describe it: it is given by the assignement $(\xi, v) \mapsto (\xi, \mu(\xi, v))$. If we fix $\xi \in F(A')$, using the fact that the map above is bijective we see that the fiber of $F(\phi)(\xi)$ are precisely the elements of the form $\xi \cdot v$ and that if $v \neq w$, then $\xi \cdot v \neq \xi \cdot w$. In other words, t_F acts in a free and transitive way on the fiber of $F(\phi)(\xi)$.

As a corollary, we obtain the following useful statement:

Corollary 3.1.5. *Assume F satisfies all the Schlessinger conditions except at most condition H_3). Then, condition H_3) is satisfied if and only if the action μ is free and transitive on the fibers of $F(\phi)$ for every small extension $\phi : A' \to A$.*

Proof. The functor F satisfies H_0) and H_1), so t_F acts on $\mathrm{Lif}(A, A', \xi)$ by the mapping $(\xi, v) \to (\xi, \xi \cdot v)$. If (ξ, v) and (ξ', v') have the same image in $F(A') \times_{F(A)} F(A')$, then $\xi = \xi'$, and $v = v'$ if and only if the action is transitive, so injectivity of α is equivalent to the freeness of the action.
Viceversa, given $(\xi, \xi') \in F(A') \times_{F(A)} F(A')$, they are both in the fiber of $F(\phi)(\xi)$, so there is a preimage (ξ, v) in $F(A' \times_A A')$ exactly when $\xi \cdot v = \xi'$, that is, when the action is transitive.
Condition H_3) is satisfied if $\alpha : F(A' \times_A A') \to F(A') \times_{F(A)} F(A')$ is a bijection for

every small extension $A' \to A$, and the discussion above shows that, given ϕ, the map α is a bijection if and only if t_F acts in a free and transitive way on the fibers of $F(\phi)$. $\square$

We end the section with a definition. We recall that we saw in 1.1.6 that $o(A/\Lambda)$ is a k-vector space.

Definition 3.1.6. *A k-vector space $o(F)$ is called an obstruction space for F if it satisfy the following properties:*

- *for every $A \in \mathcal{A}_\Lambda$ and $\xi \in F(A)$ there is a k-linear map $o(\xi) : o(A/\Lambda) \to o(F)$;*
- *a small extension $\phi : A' \to A$ is in $\ker(o(\xi))$ iff and only if ξ is in the image of $F(\phi) : F(A') \to F(A)$;*

If 0 is an obstruction space for F, we say it is unobstructed.

Notice that the obstruction space to a functor F needs not be unique: indeed, even for an unobstructed functor, any vector space is an obstruction space taking o_ξ to be the null function for every ξ. We will however see in the next section that, under some suitable hypothesis, we can find a "canonical" obstruction space for the functor H_X^Y.

3.2 Obstructions for closed subschemes

We study the concept of obstructions introduced in the previous section in the case of embedded deformations. We assume Y projective.

Definition 3.2.1. *Given an embedded deformation $\xi \in H_X^Y(A)$ of X in Y over A and a small extension $\phi : A' \to A$, we say that liftings of ξ exist locally if there is an affine* *open covering $(Y_i)_i$ of Y such that on each Y_i there exists an embedded deformation of $X_i = X \cap Y_i$ over A' that pulls back to $\xi|_{X_i}$. If this can be done for all small extensions in $\mathcal{A}_\Lambda$, we say that the embedding of X in Y is locally unobstructed.*

In other words, liftings exist locally if, on some covering, we can lift the restriction of the deformation. Since we want to find a global extension, we want to understand when these local liftings glue to a global one. The following theorem answers this question.

Theorem 3.2.2. *Let $\phi : A' \to A$ a small extension, $\xi \in H^Y_X(A)$ and assume that liftings of ξ exist locally. Then there exists an element $o_\xi(\phi) \in H^1(X, \mathcal{N}_{X/Y})$ such that $o_\xi(\phi)$ is zero if and only if there exists a lifting $\xi' \in H^Y_X(A')$ of ξ.*

Proof. We call X'_i the local liftings on X_i and $Y_{ij} = Y_i \cap Y_j$, and similarly for triple intersections. On Y_{ij} we have two liftings: $X'_i \cap Y_{ij}$ and $X'_j \cap Y_{ij}$. Both are liftings of $X_{ij} = X \cap Y_{ij}$, so by theorem 3.1.2, there is a unique element $\alpha_{ij} \in \mathcal{N}_{X/Y}(X_{ij})$ such that $(Y_{ij} \cap X'_i) \cdot \alpha_{ij} = (Y_{ij} \cap X'_j)$.
The element $(\alpha_{ij})_{ij} \in \check{C}^1((X_{ij})_{ij}, \mathcal{N}_{X/Y})$ just constructed is a Čech 1-cocycle. Indeed, since the action of the tangent space of H^Y_X on the set of liftings is free and transitive, we have $\alpha_{ik} = \alpha_{ij} + \alpha_{jk}$ on triple interesections X_{ijk}.If we choose another set of liftings $(X''_i)_i$ and $(\beta_{ij})_{ij}$ is the cocycle associated to these liftings, we show that the two cocycles differ by a coboundary. Again by theorem 3.1.2, we know that there exist $h_i \in \mathcal{N}_{X/Y}(U_i)$ such that $X'_i = X''_i \cdot h_i$. we also have that $X'_j \cap Y_{ij} = (X'_i \cap Y_{ij}) \cdot \alpha_{ij} = (X''_i \cap Y_{ij}) \cdot (h_i + \alpha_{ij})$. But $X'_j \cap Y_{ij} = (X''_j \cap Y_i j) \cdot h_j = (X''_i \cap Y_{ij}) \cdot (\beta_{ij} + h_j)$. The action is free, so $\alpha_{ij} = \beta_{ij} + h_j - h_i$, so we are done.
Moreover, the Y_i are by assumption affine, and so the X_i are affine, too. By separatedness of X, it follows that the cohomology class does not depend on the open covering we used.
If the cohomology class so constructed is zero, then we can write $\alpha_{ij} = h_i - h_j$. So,

we can multiply by the h_i's to modify the local liftings X'_i so that they are compatible. Hence, they glue to give a global lifting. Viceversa, if the local liftings define a global one, then the α_{ij} are the identity, so its cohomology class is 0. $\square$

If we assume that X is locally unobstructed, fixing A and a deformation $\xi \in H^Y_X(A)$, the previous proposition implies the existence of a function of sets

$$o_\xi : o(A/\Lambda) \to H^1(X, \mathcal{N}_{X/Y})$$

sending a small extension $\phi : A' \to A$ to the cohomology class $o_\xi(\phi)$ constructed in the proof of theorem 3.2.2.

Proposition 3.2.3. *The function $o_\xi(-)$ is a linear map.*

Proof. Consider the trivial extension $\pi : A * k \to A$, which is the zero of $o(A/\Lambda)$. There is a morphism $i : A \to A * k$ which gives the identity of A when composed with π. Then this implies that $i_\sharp(\xi) \in H^Y_X(A * k)$ is a lifting of ξ, so $o_\xi(\pi)$ is zero by theorem 3.2.2.

Take now two small extensions $\phi : A' \to A$, $\psi : A'' \to A$, such that $o_\xi(A')$ (resp. $o_\xi(A'')$) is represented by (α_{ij}) (resp. β_{ij}). The sum of ϕ and ψ was defined to be the small extension $(A' \times_A A'')/I$, where I is the kernel of the sum morphism $k^2 \to k$. Take now local liftings $(X'_i)_i$ (resp.$(X''_i)_i$)) over A' (resp. A'') (notice that, a priori these two local liftings are not defined locally on the same open covering, but we can take a refinement of both coverings and hence suppose that the open sets underlying X'_i and X''_i are the same for every i). Then, for every i we have an element of $H^{Y_i}_{X_i}(A') \times_{H^{Y_i}_{X_i}} (A) H^{Y_i}_{X_i}(A'')$. So, by theorem 2.2.3 X'_i and X''_i define a deformation of X_i over $A' \times_A A''$ that pulls back to X_i, i.e. a set of local liftings of X over $A' \times_A A''$.

As done in theorem 3.2.2, we can now assign an element of $H^1(X, \mathcal{N}_{X/Y}) \otimes_k k^2$ corresponding to these local liftings. In this case, there is an additional factor $- \otimes_k k^2$ because $A' \times_A A''$ is not a small extension over A, but has kernel of length 2. Indeed, just as done in theorem 3.2.2, one can show that if X is locally unobstructed and if $B \to A$ has kernel of length n, there is an element in $H^1(X, \mathcal{N}_{X/Y}) \otimes_k k^n$ representing the local liftings over B, with the porperty of being 0 if and only if the local liftings glue to a global one. Moreover, the morphism $\pi^1_\sharp$ projecting $H^1(X, \mathcal{N}_{X/Y}) \otimes_k k^2$ on its first (resp. second) component is the morphism that corresponds to the projection $\pi^1 : A' \times_A A'' \to A'$ (resp. the projection π^2 to A''), in the following sense. If $\bar{X}_i$ are local liftings on $A' \times_A A''$ and γ_{ij} the coboundary associated to these liftings, then $\pi^1_\sharp(\gamma_{ij})$ is the coboundary associated to the local liftings $\pi^1_\sharp(\bar{X}_i) = X'_i$, that is, $\pi^1_\sharp(\gamma_{ij}) = \alpha_{ij}$ (resp. $(\pi^2_\sharp(\gamma_{ij}) = \beta_{ij}$). this follows again by freeness of the action of the tangent space on local liftings. It follows that $(\gamma_{ij}) = (\alpha_{ij}, \beta_{ij}) \in H^1(X, \mathcal{N}_{X/Y}) \otimes_k k^2$. Finally, if f is the projection $A' \times_A A'' \to (A' \times_A A'')/I$, there is a morphism $f_\sharp : H^1(X, \mathcal{N}_{X/Y}) \otimes_k k^2 \to H^1(X, \mathcal{N}_{X/Y})$ that assigns (γ_{ij}) to the cohomology class of the small extension $\phi + \psi$, done exactly as above. $f_\sharp$ is the morphism induced by the sum morphism $k^2 \to k$ (this follows again from freeness of the action of t_F), so $f_\sharp(\gamma_{ij}) = \alpha_{ij} + \beta_{ij}$, so we conclude that $o_\xi(\phi + \psi) = o_\xi(\phi) + o_\xi(\psi)$.
Finally, let $x \in k^*$ and $\phi \in o(A/\Lambda)$ a small extension. Arguing as above and using again the freeness of the action of the tangent space, we find that $x \cdot o_\xi(\phi) = o_\xi(x \cdot \phi)$. $\square$

We can rephrase the previous results in terms of the obstruction spaces introduced in definition 3.1.6:

Proposition 3.2.4. *Let X locally unobstructed in Y. Then the space $H^1(X, \mathcal{N}_{\mathcal{X}/\mathcal{Y}})$ is an obstruction space for the functor H^Y_X.*

Proof. It follows from theorem 3.2.2 and 3.2.3. □

Theorem 3.2.2 is interesesting, albeit not very useful if we don't know whether the embedding $X \subset Y$ admits local liftings or not. The following results, whose proofs can be found in [Har10] give examples of cases in which the assumptions of theorem 3.2.2 are satisfied, and which we will use in the third part of the chapter.

First, a definition.

Definition 3.2.5. *A scheme X is Cohen-Macaulay if all the local rings $\mathcal{O}_{X,x}$ are Cohen-Macaulay.*

Theorem 3.2.6 ([Har10, Thm. 8.3])**.** *Let Y a smooth scheme, and $X \subset Y$ a closed subscheme of codimension 2 which is also Cohen-Macaulay Then X is locally unobstructed in Y.*

Another useful instance in which we have local unobstructedness is when X is a locally complete intersection.

Definition 3.2.7. *A closed subscheme X of a smooth variety Y is a a local complete intersection in Y if the sheaf ideal $\mathcal{I}_X$ can be locally generated by $\mathrm{Codim}_Y(X)$ elements at every point. A closed subscheme $X \subset \mathbb{P}^n$ is a complete intersection if the associated ideal $I_X \subset k[x_0, \dots, x_n]$ can be generated by $\mathrm{Codim}_{\mathbb{P}^n}(X)$ elements.*

We notice that every global complete intersection is trivially a local one.

Theorem 3.2.8 ([Har10, Thm. 9.2])**.** *Let $X \subset \mathbb{P}^n$ a locally complete intersection scheme. Then X is locally unobstructed in $\mathbb{P}^n$.*

Every smooth closed subscheme of a smooth scheme Y is a local complete intersection in it, see [Har77, II Thm. 8.17]. We have thus the following:

Corollary 3.2.9. *If $X \subset Y$ is a smooth closed subscheme of a smooth variety, then X is locally unobstructed in Y.*

3.3 Geometric applications

In this section we are interested in showing some simple examples of how the theory of the obstructions can be used to prove some results about Hilbert schemes. So, for the remainder of the chapter we assume that $\Lambda = k$. We use the results of the previous sections to examine in greater detail some Hilbert schemes. We start by exploring the concept of formal smoothness. To do so, we will need the following criterion for smoothness.

Theorem 3.3.1 (Finch's theorem)**.** *Let X be an algebraic scheme such that for every small extension $\phi : A' \to A$ and for every morphism $f : \operatorname{Spec} A \to X$ there is a morphism $g : \operatorname{Spec} A' \to X$ that lifts f, that is $f = g \cdot \operatorname{Spec}(\phi)$. Then X is smooth.*

The condition of theorem 3.3.4 is a special case of the *formal criterion for smoothness* and is usually depicted with the following diagram

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r, "f"] \arrow[d, "\operatorname{Spec}(\phi)"'] & X \arrow[d] \\ \operatorname{Spec} A' \arrow[r] \arrow[ur, dashed, "g"] & \operatorname{Spec} k \end{tikzcd}$$

In the proof of the theorem, we will need the following two propositions.

Proposition 3.3.2. *Let B be a Noetherian local ring, and $\hat{B}$ be its completion. Then B is regular if and only if $\hat{B}$ is regular.*

A proof can be found in https://stacks.math.columbia.edu/tag/07NY.
The second one is:

Lemma 3.3.3. *Let A be a Noetherian local k-algebra with residue field k and maximal ideal $\mathfrak{m}$. If $f : A \to A$ is a k-morphism inducing an isomorphism $\bar{f} : A/\mathfrak{m}^2 \to A/\mathfrak{m}^2$, then f is an isomorphism.*

Proof. Let I the ideal generated by the images of the generators m_i of $\mathfrak{m}$. If $\bar{f}$ is an isomorphism, then $I/\mathfrak{m}^2 \simeq \mathfrak{m}/\mathfrak{m}^2$, so $\mathfrak{m} = \mathfrak{m}^2 + I$. So, $\mathfrak{m}/I = \mathfrak{m}^2/I$. Applying Nakayama's lemma, we find that $I = \mathfrak{m}$, i.e. $f(\mathfrak{m}) = \mathfrak{m}$. Moreover, since the residue field of A is k, every element of A can be written in a unique way as a sum of an element of k (which is a subring of A, since A is a k-algebra) and of $\mathfrak{m}$. Indeed, if $i : k \to A$ is the structure morphism and $\pi : A \to k$ the projection, if $a \in A$, $a = i(\pi(a)) + (a - i(\pi(a)))$.
Being a k-morphism, f is the identity on elements of the form $i(t)$, $t \in k$. Since we know that $f(\mathfrak{m}) = \mathfrak{m}$, we conclude that f is surjective.
To prove injectivity, note that A is an $A[x]$-module by $g(x) \cdot a = g(f)(a)$. Surjectivity of f implies that $A = (x) \cdot A$, so by Nakayama's lemma there exists $g \in (x)$ such that $g \mod (x) = 1$ and $g \cdot A = 0$. So, let $g(x) = 1 + xp(x)$, and let $a \in \ker(f)$. So, $0 = g \cdot a = a + xp(x) \cdot a = a$, from which we conclude. □

We now prove theorem 3.3.1.

Proof. We have to prove that for every closed point x of X, the local ring $\mathcal{O}_{X,x}$ is regular. This statement is local, so we reduce to the affine case. So, we may assume to have a local k-algebra B that is the localisation of a k-algebra of finite type and having k as residue field, with the property that for every solid diagram

$$\begin{tikzcd} k \arrow[r] \arrow[d] & A' \arrow[d, "\phi"] \\ B \arrow[r] \arrow[ur, dashed] & A \end{tikzcd}$$

where ϕ is a surjection of k-Artinian algebras, there is a dashed arrow that makes the diagram commute.

Let $b_1, \ldots, b_n$ minimal generators of the maximal ideal $\mathfrak{m}$ of B. There are compatible surjective morphisms of k-algebras $r_i : k[x_1, \ldots, x_n]/(x_1, \ldots, x_n)^i = R_i \to B/\mathfrak{m}^i$ obtained by mapping $x_j \mapsto b_j$. Taking the inverse limit, this yields a surjective morphism

$$r : k[[x_1, \ldots, x_n]] \to \hat{B}$$

Call $\mathfrak{p}$ the ideal $(x_1, \ldots, x_n) \subset k[x_1, \ldots, x_n]$. By minimality of the b_i's, r_2 restricts to an isomorphism of the submodules $\mathfrak{p}/\mathfrak{p}^2$ and $\mathfrak{m}/\mathfrak{m}^2$. Both rows of the following commutative diagram are exact, so by the $5-$lemma we conclude that r_2 is an isomorphism.

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{p}/\mathfrak{p}^2 \arrow[r] \arrow[d] & R_2 \arrow[r] \arrow[d, "r_2"] & k \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathfrak{m}/\mathfrak{m}^2 \arrow[r] & B/\mathfrak{m}^2 \arrow[r] & k \arrow[r] & 0
\end{tikzcd}$$

We now inductively construct a morphism $p : \hat{B} \to k[[x_1, \ldots, x_n]]$ as follows. There is a morphism $p_2 : B \to R_2$ by taking the composition of the projection onto $B/\mathfrak{m}^2$ and then composing with the inverse of r_2. We notice now that, if $i \geq 2$, the R_i's are Artinian k-algebras (they are Noetherian and of dimension 0), so using the formal smoothness property of B, we lift p_2 inductively to compatible maps $p_i : B \to R_i$

$$\begin{tikzcd}
k \arrow[r] \arrow[d] & R_{i+1} \arrow[d] \\
B \arrow[r, "p_i"'] \arrow[ur, dashed, "p_{i+1}"] & R_i
\end{tikzcd}$$

In this way, we assemble the map p, and we now consider the composition

$$k[[x_1, \ldots, x_n]] \to \hat{B} \to k[[x_1, \ldots, x_n]]$$

This composition, which we call ψ, has the property that, by construction of the maps, the induced endomorphism $R_2 \to R_2$ is an isomorphism. Then, applying lemma 3.3.3, we find that ψ is an isomorphism, so r must be injective. We already knew that r was surjective, so we conclude that it is an isomorphism. So $\hat{B}$ is regular, and hence B is, too, by proposition 3.3.2. □

Recall the notation we introduced in chapter 2.3: given a scheme X, if $A \in \mathcal{A}$, we let $X(A)_x$ denote the set of morphisms $\operatorname{Spec} A \to X$ having image x. This allows us to restate theorem 3.3.4 in this form:

Theorem 3.3.4. *Let X be an algebraic scheme and $x \in X(k)$. Then X is regular at x if and only if $X(-)_x$ is unobstructed.*

This reformulation of theorem 3.3.4 will be more useful to us, because of the theory we developed for the obstructions of embedded deformations. We now use it to show some examples, beginning with a definition.

Definition 3.3.5. *A subscheme $X \subset \mathbb{P}^n$ has length l if the Hilbert polynomial of X is the constant polynomial l.*

Since the Hilbert polynomial of a closed subscheme $X \subset \mathbb{P}^n$ has degree equal to the dimension of X, a subscheme of length l must be zero dimensional, and a 0-dimensional subscheme of $\mathbb{P}^n$ of finite length is a finite union of points

$$Z = \amalg_z \operatorname{Spec} \mathcal{O}_{Z,z}$$

where $\mathcal{O}_{Z,z}$ is an Artinian k-algebra of length n_z, such that $\sum_z n_z = l$. The Hilbert schemes $\mathrm{Hilb}^k_{\mathbb{P}^n}$ are also called the Hilbert schemes of l-points.

Proposition 3.3.6. *The Hilbert scheme of subschemes of length k in $\mathbb{P}^2$ is smooth.*

Proof. X is zero dimensional, and since the depth of a local ring cannot be greater than its dimension, all of its local rings are Cohen-Macaulay. So, by theorem 3.2.6, X is locally unobstructed in $\mathbb{P}^2$. Moreover, since X is zero dimensional, $H^1(X, \mathcal{F})$ vanishes for every coherent sheaf on X, and so $H_X^{\mathbb{P}^2}$ is unobstructed by theorem 3.2.2. Hence, applying theorem 3.3.4 we conclude that the Hilbert scheme is smooth. $\square$

Example 3.3.1. Using the Hilbert schemes of points we can show that the local unobstructedness hypothesis in theorem 3.2.2 is necessary. We give here an example a Cohen-Macaulay subscheme of codimension 3 in $\mathbb{P}^3$ that is obstructed. Let $X = \operatorname{Spec} k[x, y, z]/\mathfrak{m}^2$, where $\mathfrak{m} = (x, y, z)$ and consider X as a subscheme of $\mathbb{P}^3$. X is a subscheme of length 4. It can be shown that the point corresponding to X lies in an irreducible component of $\mathrm{Hilb}^4_{\mathbb{P}^3}$ of dimension 12, see [Fan+05, Section 7.2]. However, a direct computation shows that $h^0(\mathcal{N}_{X/Y}) = 18$, so X does not correspond to a smooth point of $\mathrm{Hilb}^4_{\mathbb{P}^3}$, even though $H^1(\mathcal{N}_{X/Y}) = 0$.

We can also finally prove a result promised in 1.3.5, that is the smoothness of the Hilbert scheme of plane curves.

Proposition 3.3.7. *The Hilbert scheme of plane curves $\mathrm{Hilb}^{curves}_{\mathbb{P}^2}$ is smooth.*

Proof. Let C be a plane curve. Since plane curves are hypersurfaces in $\mathbb{P}^2$, they are global complete intersections and hence lci, so by theorem 3.2.8 locally unobstructed. Moreover, we have that $I_C \simeq \mathcal{O}(-\deg C)$. Hence, the normal sheaf of C is the $\mathcal{O}_C$-dual of $i^*\mathcal{O}(-\deg C)$, and we obtain $\mathcal{N}_{C/\mathbb{P}^2} = \mathcal{O}_C(\deg C)$.

Moreover, there is a short exact sequence of sheaves

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(\deg C) \longrightarrow i_*\mathcal{O}_C(\deg C) \longrightarrow 0$$

obtained by tensoring the exact sequence defining $i_*\mathcal{O}_C$ by $\mathcal{O}(\deg C)$. Taking the associated long exact sequence in cohomology, we find that

$$H^1(\mathbb{P}^2, i_*\mathcal{O}_C(\deg C)) = H^1(C, \mathcal{O}_C(\deg C)) = 0$$

So there are no obstructions to liftings, and we conclude by using theorem 3.3.4. □

In all these examples, we used that a subscheme is unobstructed if $H^1(X, \mathcal{N}_{X/Y})$ is trivial. To end the chapter, we show that it not necessary that this cohomology group is zero for X to be unobstructed in Y. Indeed, if X is rigid in Y (see definition 1.2.8) then clearly it is unobstructed, and we have the following simple criterion for rigidity.

Proposition 3.3.8. *Assume that $H^0(X, \mathcal{N}_{X/Y}) = 0$. Then X is rigid in Y.*

Proof. We saw in theorem 3.1.2 that, under the Schlessinger conidtions, the tangent space of an Artin ring functor acts in a free and transitive way on the liftings. We use this to prove by induction that $H_X^Y(A)$ is a singleton for every Artin ring A. We do induction on the length n of the ring.
We start with $n = 2$. The only k-algebra that is Artinian and of length 2 is $k[\varepsilon]$, and so the base step is true by the assumption on the tangent space.
For the $(n+1)$-step, quotient A to get an algebra $\tilde{A}$ of length n. Notice that $A \to \tilde{A}$ is a small extension. By assumption, there is a single embedded deformation over $\tilde{A}$, the trivial one. Any embedded deformation over A is a lifting of this trivial

deformation, so $H^0(X, \mathcal{N}_{X/Y})$ acts on $H_X^Y(A)$ in a free and transitive way. But the tangent space is trivial, so we conclude that $H_X^Y(A)$ can have only one element. $\square$

Example 3.3.2. Let C be a smooth projective curve of genus $g \geq 2$. By separatedness, the diagonal morphism $\Delta : C \to C \times C = S$ is a closed immersion, and we identify its image with C. If $\mathcal{I}$ is the ideal sheaf defining this immersion, then the normal sheaf $\mathcal{N}_{C/S}$ is the dual of $\mathcal{I}/\mathcal{I}^2$. The latter is by definition the contanget sheaf $\Omega_{C/k}$, so we find that $\mathcal{N}_{C/S}$ is the tanget sheaf $\mathcal{T}_C$ of C. Serre duality and Riemann-Roch imply that $h^1(C, \mathcal{T}_C) = 3g - 3 \neq 0$. However, C is unobstructed in S. Indeed, using again Riemann-Roch we find that $H^0(C, \mathcal{T}_C) = 0$, so by the proposition above, C is rigid in S and hence unobstructed.

Chapter 4

Formal deformations and Schlessinger's theorem

This chapter is less concerned with the study of deformations of embedded deformations, and focuses on the topic of formal deformations. When we defined infinitesimal deformations, we stated that they were the building blocks of formal deformations, and in this chapter we see how that is. We explain here why formal deformations are relevant.

Suppose that we are interested in the deformation of a projective variety X over some curve, for example the affine line. Some such examples are shown in section 1.2. We could, however, not have a deformation over $\mathbb{A}^1$ to start with, but we may be able to construct iteratively deformations $\mathcal{X}_n$ over $\operatorname{Spec}(k[x]/(x^n))$ which are compatible, i.e. the sequence of infinitesimal deformations is coherent. This would correspond, in the language of complex geometry, to taking the Taylor expansion of order n of the equations defining X, and taking the limit we would get a formal series. We could then consider the region (supposing it exists) where said series converge and obtain a deformation of X over some open subset of $\mathbb{C}$.

In the language of algebra, taking the limit means considering the deformations $\{\mathcal{X}_n\}$ as a formal object, i.e. as an element of an inverse limit of a tower of sets.

One geometric way of describing the formal deformation so obtained is as a formal scheme over the formal spectrum of $k[[x]]$. The problem of "convergence" (called, in this context, *algebraizability*) is however more subtle. Indeed, already in the complex-algebraic sense, we would need to ask that the series converges to an algebraic function, a very hard condition. Moreover, if in the complex-algebraic sense we could see the formal deformation as a deformation over $\mathbb{C}[[x]]$, in the algebro-geometric problem this is not necessarily true. A overview of this phenomena can be found in [Ser06, Section 2.5].
In this chapter we introduce the notion of formal deformation, and then proceed to define miniversal and universal deformations, which allow us to end the chapter with the statement of the important Schlessinger theorem.

4.1 Formal elements

Let Λ a complete Noetherian local ring with residue field k.

Definition 4.1.1. *The category $\hat{\mathcal{A}}_\Lambda$ is the category having local complete Λ-algebras with residue field k as objects and local morphisms of Λ-algebras as morphisms.*

In the following, if $R \in \hat{\mathcal{A}}_\Lambda$, we will denote the maximal ideal of R by $\mathfrak{m}_R$.
Given a functor of Artin rings $F : \mathcal{A}_\Lambda \to \mathrm{Set}$, we can upgrade it to a functor $\widehat{F} : \hat{\mathcal{A}}_\Lambda \to \mathrm{Set}$ in the following way. Given $R \in \hat{\mathcal{A}}_\Lambda$, consider the tower of projections

$$\cdots \to R/\mathfrak{m}_R^n \to R/\mathfrak{m}_R^{n-1} \to \dots$$

Then let $\widehat{F}(R)$ be the inverse limit of the system

$$\cdots \to F(R/\mathfrak{m}_R^n) \to F(R/\mathfrak{m}_R^{n-1}) \to \dots$$

where the maps are the ones induced by F.
Given a morphism $f : R \to S$ in $\hat{\mathcal{A}}_\Lambda$, there are induced morphisms $f_n : R/\mathfrak{m}_R^n \to S/\mathfrak{m}_S^n$, which in turn yield yield compatible morphisms $F(f_n)$ once we apply F. Taking the direct limit then gives a morphism $\widehat{F}(f) : \widehat{F}(R) \to \widehat{F}(S)$, and properties of limits imply the functoriality of our construction.

Remark 4.1.1. Strictly speaking, the functoriality of $\widehat{F}$ is not guaranteed, since the limit of a tower of sets is not unique, but only unique up to a unique isomorphism, so then $\widehat{F}$ in truth would be a pseudofunctor, see https://ncatlab.org/nlab/show/pseudofunctor. However, we shall always use a unique "canonical" model for the limit of a tower, that is the set of coherent sequences, and hence treat $\widehat{F}$ as a true functor.

Remark 4.1.2. Since the maximal ideal of an Artinian ring is nilpotent, it follows that if $A \in \mathcal{A}_\Lambda$ then $\widehat{F}(A) = F(A)$, and the same goes for morphisms. So, $\widehat{F}$ restricted to the subcategory $\mathcal{A}_\Lambda$ is the functor F we started with.

Remark 4.1.3. From the properties of complete rings, it is readily checked that $h_{R/\Lambda}$ satisfies all the Schlessinger conditions. In particular, it is then well defined the tangent space to $h_{R/\Lambda}$.

Example 4.1.1. If $R \in \hat{\mathcal{A}}_\Lambda$, we defined in example 1.1.1 what it is the functor prorepresented by R: it is $\mathrm{Hom}_\Lambda(R, -)$, which we will from now on denote by $h_{R/\Lambda}$. By basic properties of completions, it follows that $\widehat{h_{R/\Lambda}}$ is simply $\mathrm{Hom}_\Lambda(R, -)$. In particular, if A is Artinian and n is such that $\mathfrak{m}_A^n = 0$, then $h_{(R/\mathfrak{m}^n)/\Lambda}(A) \to h_{(R/\mathfrak{m}^{n+1})/\Lambda}(A)$ is a bijection.

Definition 4.1.2. *An element $\hat{u} \in \widehat{F}(R)$ is called a formal element for R.*

Recall that, for an object X of some category $\mathcal{C}$ and a functor $F : \mathcal{C} \to \mathrm{Set}$, the Yoneda lemma yields a bijection between $F(X)$ and natural transformations

$\mathrm{Hom}_{\mathcal{C}}(X, -) \to F$. In our case, we cannot apply the Yoneda lemma to prorepresentable functors since they are not necessarily represented by objects of $\mathcal{A}_\Lambda$. We can, however, prove the following formal version of the lemma.

Lemma 4.1.3 (Formal Yoneda lemma). *Given $R \in \hat{\mathcal{A}}_\Lambda$, there is a bijection between natural transformations $h_{R/\Lambda} \to F$ and elements of $\widehat{F}(R)$.*

For notation's sake, in the following proof and only there, we write h_R instead of $h_{R/\Lambda}$.

Proof. Let $\hat{u} \in \widehat{F}(R)$. Then $\hat{u}$ is the datum of a coherent sequence $(u_n)_n$ with $u_n \in F(R/\mathfrak{m}^{n+1})$. Let k^n be the natural transformation $k^n : h_{R/\mathfrak{m}^n} \to h_{R/\mathfrak{m}^{n+1}}$ induced by the projection.
By the Yoneda lemma, each u_n defines a natural transformation $\phi^n : h_{R/\mathfrak{m}^{n+1}} \to F$. The coherence of the sequence implies that the following diagram commutes

$$\begin{tikzcd} h_{R/\mathfrak{m}^n} \arrow[r, "k^n"] \arrow[dr, "\phi_{n-1}"'] & h_{R/\mathfrak{m}^{n+1}} \arrow[d, "\phi^n"] \\ & F \end{tikzcd}$$

By example 4.1.1, k^n_A is a bijection for a sufficiently large n, so there is an isomorphism $g_A : h_{R/\mathfrak{m}^n}(A) \to h_R(A)$. Then, we define $\phi_A : h_R(A) \to F(A)$ as the composition

$$\begin{tikzcd} h_R(A) \arrow[r, dashed, "\phi_A"] \arrow[d, "g_A^{-1}"'] & F(A) \\ h_{R/\mathfrak{m}^n}(A) \arrow[ur, "\phi^n_A"'] & \end{tikzcd}$$

To check that it gives a natural transformation $\phi : h_R \to F$, given a morphism $f : A \to B$ in $\mathcal{A}_\Lambda$, by taking n sufficiently large it immediately follows from

naturality of the ϕ^n's.
Conversely, given a natural tranformation $\phi : h_R \to F$, let $u_n \in F(R/\mathfrak{m}^{n+1})$ the image of the canonical projection $R \to R/\mathfrak{m}^{n+1}$ under $\phi_{R/\mathfrak{m}^{n+1}}$. Naturality of ϕ implies that the sequence is coherent, so it defines an element of $\widehat{F}(R)$.
Using the standard Yoneda lemma, once can now show that the constructions made above are indeed inverse to each other. □

Given R as above and $\hat{u} \in \widehat{F}(R)$, we say that $(R, \hat{u})$ is a *formal couple* for F. In view of remark 4.1.3, we call the differential of the natural transformation idnuced by $\hat{u}$ the *characteristic map* of $\hat{u}$, and we denote it by $d\hat{u} : t_{h_{R/\Lambda}} \to t_F$.

Definition 4.1.4. *A formal couple $(R, \hat{u})$ is called universal if the induced morphism $h_{R/\Lambda} \to F$ is an isomorphism of functors.*

It is clear from the definition that, if $(R, \hat{u})$ is a universal formal couple, then F is prorepresented by R.

Consider now two functors of Artin rings F and G, and let $f : F \to G$ be a morphism between them. Let $\phi : A \to B$ a morphism in $\mathcal{A}_\Lambda$ Then, the diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{f_A} & G(A) \\ \downarrow{\scriptstyle F(\phi)} & & \downarrow{\scriptstyle G(\phi)} \\ F(B) & \xrightarrow{f_B} & G(B) \end{array}$$

yields a natural map

$$\beta : F(A) \to F(B) \times_{G(B)} G(A)$$

Definition 4.1.5. *A morphism f as above is called smooth if the map β is a surjection whenever $\phi : A \to B$ is a small extension. If G is the singleton functor and f is smooth, F is said to be a smooth functor.*

Here, by singleton functor, we mean a functor G such that $G(A) = *$ for every object A. Notice that, by remark 1.1.2, if a morphism $f : F \to G$ is smooth, then β is also surjective whenever $A \to B$ is surjective.

A functor F is smooth if and only if the map $F(A) \to F(B)$ is surjective whenever $A \to B$ is small, so F is smooth if and only if it is unobstructed. This ties well with the interpretation of unobstructedness we found in theorem 3.3.4. Moreso, the term *smooth* for a morphism of functors has its origin in the following porperty of morphism of rings.

Definition 4.1.6. *Consider a ring morphism $\phi : R \to S$. It is called a formally smooth morphism if, for every $R-$algebra B, square-zero ideal $I \subset B$ and every solid diagram*

$$\begin{tikzcd} S \arrow[r] \arrow[dr, dashed] & B/I \\ R \arrow[u, "\phi"] \arrow[r] & B \arrow[u, "\pi"'] \end{tikzcd}$$

where π is the projection, there is a dashed arrow making the diagram commute.

When translated in the language of schemes, the notion of formal smoothness recovers the usual one. Indeed, if we ask that $f : X \to Y$ is locally of finite presentation and has the infinitesimal lifting property for every first order thickening of affine Y-schemes, then f is smooth. A proof can be found in https://stacks.math.columbia.edu/tag/02H6.

In the case when R is Noetherian, the situation is simpler. Indeed, the following fact, which we also do not prove, holds:

Proposition 4.1.7. *If R is Noetherian, a morphism of finite presentation $R \to S$ is formally smooth if the infinitesimal lifting property holds for small extensions of Artinian rings.*

A reference for the proof is https://stacks.math.columbia.edu/tag/02HX. Also, notice that we proved this in theorem 3.3.4 in the case when $X \to \operatorname{Spec} k$ is structure morphism.

If $f : R \to S$ is a morphism of Noetherian local Λ-algebras with residue field k, there is an induced morphism of Artin rings $h_f : h_{S/\Lambda} \to h_{R/\Lambda}$ (we previously defined the fucntors $h_{R/\Lambda}$ only for complete local Noetherian rings, but their definition can be extended in an obvious way for Noetherian algebras with residue field k).

Proposition 4.1.8. *If R is Noetherian, a morphism of finite presentation $R \to S$ is formally smooth if and only if h_f is smooth.*

Proof. This is just playing around with the definitons. h_f is smooth if and only if, givan any small extension $A \to B$, the map

$$h_{S/\Lambda}(A) \longrightarrow h_{R/\Lambda}(A) \times_{h_{R/\Lambda}(B)} h_{S/\Lambda}(B)$$

is surjective. But this happens iff and only if f has the infinitesimal lifting property. $\square$

4.2 Formal deformations

In this section R denotes an object of $\in \hat{\mathcal{A}}_\Lambda$ and A an object of $\mathcal{A}_\Lambda$. All morphisms will be in the appropriate category.

Definition 4.2.1. *A formal deformation of X over R is an element $\bar{\mathcal{X}}$ of $\widehat{Def_X}(R)$.*

We spell out in detail the meaning of definition 4.2.1. A formal deformation $\bar{\mathcal{X}}$ is the datum, for every n, of a deformation $\mathcal{X}_n$ of X over $R/\mathfrak{m}^{n+1}$, such that the pullback of $\mathcal{X}_n$ along $\mathrm{Spec}(R/\mathfrak{m}^n) \to \mathrm{Spec}(R/\mathfrak{m}^{n+1})$ is isomorphic to $\mathcal{X}_{n-1}$. Also, given a morphism $f : R \to A$, $\bar{\mathcal{X}}$ induces a deformation $\mathcal{X}$ on A. We can use the explicit construction in the proof of the Formal Yoneda Lemma to determine $\mathcal{X}$. Indeed, let n such that $\mathfrak{m}_A^{n+1} = 0$. Then f factors as $R/\mathfrak{m}^{n+1} \to A$. Then $\mathcal{X}$ is the pullback along this morphism of $\mathcal{X}_n$.

We introduced in the previous section the notion of formal couple. We say then that a formal deformation $\bar{\mathcal{X}}$ over R is universal if it forms a universal formal couple. Univeral formal couple, however, do not usually exist. They do for the Hilbert local functor H_X^Y if Y is projective, since we saw in remark 2.3.1 that H_X^Y is then prorepresentable, but they are not guaranteed to exist for the functors Def_X and Def'_X. So we introduce two weaker notions of universality.

Definition 4.2.2. *A formal deformation $\bar{\mathcal{X}}$ over R is versal if the induced morphism $h_{R/\Lambda} \to \mathrm{Def}_X$ is smooth. It is miniversal if it is versal and moreover, the differential $d\bar{\mathcal{X}}$ is an isomorphism.*

The definition of versal and miniversal element for a functor of Artin rings is analogous.

Again, we explicitly spell out the property of versality for a formal couple $(R, \bar{\mathcal{X}})$. Given a small extension $A' \to A$ and a morphism $f : R \to A$, we have induced a deformation $\mathcal{X}$ of X over A. Let $\mathcal{X}'$ a deformation over A' that lifts $\mathcal{X}$. Then, the formal couple is smooth if there is a morphism $g : R \to A'$ such that

$$\begin{tikzcd} R \arrow[r, "g"] \arrow[dr, "f"'] & A' \arrow[d] \\ & A \end{tikzcd}$$

commutes and g induces $\mathcal{X}'$ on A'.

With all the work done so far, we can finally state Schlessinger's theorem:

Theorem 4.2.3. *[Sch68, Thm. 2.11] Suppose F is a functor of Artin rings satisfying condition H_0). Then F has a miniversal element if and only it satisfies conditions H_1), H_2) and H_4). If F has a miniversal element, it then has a universal element if and only if it satisfies condition H_3).*

The Schlessinger's theorem is extremely useful in proving the existence of miniversal deformations for various deformation problems. For example, one can prove a result similar to theorem 2.2.3 for the functor Def_X, except that this functor does not usually satisfy condition H_3). So, if X is such that the tangent space to Def_X is finite dimensional (for example if X has isolated singularities), then Schlessinger's theoerm guarantees the existence of a miniversal deformation. It is, however, harder to verify if a functor satisfies condition H_3), but with the work done in chapter 3 we find the following equivalent criterion.

Proposition 4.2.4. *If F has a miniversal element, then F has a universal element if and only if for every small extension $\phi : A' \to A$ the action of t_F is free and transitive on the nonempty fibers of $F(\phi)$.*

Proof. Combine theorem 4.2.3 and 3.1.5. □

As a corollary of Schlessinger's theorem, we can give an alternate proof of the prorepresentability of H_X^Y when Y is projective, see remark 2.3.1. Indeed, we saw in corollary 2.2.7 that, if Y is projective, then H_X^Y satisfies all the Schlessinger conditions. This then, by Schlessinger's result, implied prorepresentability.

Bibliography

[Fan+05] B. Fantechi et al. *Fundamental algebraic geometry. Grothendieck's FGA explained*. Mathematical Surveys and Monographs 123. American Mathematical Society, Providence, RI, 2005, pp. x+339. ISBN: 0-8218-3541-6.

[Gro62] A. Grothendieck. *Fondements de la géométrie algébrique [Extraits du Séminaire Bourbaki 1957–1962]*. 1962.

[Har77] R. Hartshorne. *Algebraic Geometry*. Graduate Texts in Mathematics 52. Springer-Verlag, New York-Heidelberg, 1977, pp. xvi+496. ISBN: 0-387-90244-9.

[Har10] R. Hartshorne. *Deformation theory*. Graduate Texts in Mathematics 257. Springer, New York, 2010, pp. viii+234. ISBN: 978-1-4419-1595-5.

[Mat86] H. Matsumura. *Commutative Ring Theory, Translated from the Japanese by M. Reid. Second edition*. Cambridge Studies in Advanced Mathematics 8. Cambridge University Press, Cambridge, 1986, pp. xiv +320. ISBN: 0-521-25916-9.

[MFK94] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric Invariant Theory, Third Edition*. Ergebnisse der Mathematik und ihrer Grenzgebiete (2) 34. Springer-Verlag, Berlin, 1994, pp. xiv+292. ISBN: 3-540-56963-4.

[Sch68] M. Schlessinger. "Functors of Artin rings". In: *Transactions AMS* 130 (1968), pp. 208–22.

[Ser06] E. Sernesi. *Deformations of Algebraic Schemes*. Grundlehren der Mathematischen Wissenschaften 334. Springer-Verlag, Berlin, 2006, pp. xii+339. ISBN: 978-3-540-30608-5; 3-540-30608-0.

[WG10] T. Wedhorn and U. Gortz. *Algebraic Geometry I*. Springer Studium Mathematik-Master. Springer Spektrum, Wiesbaden, 2010, pp. vii+625. ISBN: 978-3-658-30732-5.